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Duality Theory for Variable Costs in Joint Production*

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Abstract: Duality methods for incomplete systems of consumer demand equations are adapted to the dual structure of variable cost functions in joint production. This allows the identification of necessary and sufficient restrictions on technology and cost so that the conditional factor demands can be written as functions of input prices, fixed inputs, and cost. These are observable when the variable inputs are chosen and committed to production, hence the identified restrictions allow *ex ante* conditional demands to be studied using observable data. This class of production technologies is consistent with all von Neumann-Morgenstern utility functions when *ex post* production is uncertain.

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Analysis of multi-product behavior of firms is common in agricultural economics. Techniques of analysis might be based on the distance or production functions, or profit, revenue, or cost functions (Färe and Primont 1995; Just, Zilberman, and Hochman 1988; Shumway 1983, Lopez 1983; Akridge and Hertel 1986). There is a large literature on functional structure and duality that helps guide empirical formulations and testing based on concepts of non-jointness and separability (Lau 1972, 1978; Blackorby, Primont and Russell 1977; Chambers 1984). For example, separability in some partition of inputs or outputs often results in separability in a similar partition of prices so long as aggregator functions are homothetic (e.g., Blackorby, Primont and Russell 1977; Lau 1978). This allows a researcher to test hypotheses about the structure of technology using cost or profit functions (Shumway 1983). Similarly, the implications of non-jointness often reduce to some form of additivity (Hall 1973; Kohli 1983). Such restrictions on technology guide empiricists as they think about aggregation based on functional structure.

In this short paper an issue of functional structure is considered which is somewhat non-standard but useful to empirical work. The question considered is: when can conventional short-run cost minimizing factor demands a) $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z})$ be written as b) $\boldsymbol{x} = \tilde{\boldsymbol{X}}(\boldsymbol{w}, c, \boldsymbol{z})$, where $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$ are vector valued functions, $\boldsymbol{w}$ the corresponding vector of input prices, $\boldsymbol{z}$ is a vector of fixed inputs, $\boldsymbol{y}$ is a vector of outputs, and c is cost? More precisely, what restrictions on technology, and hence costs, imply that the conditional factor demands can be written as functions of input prices, fixed inputs, and cost rather than the more standard representation in a)?

Interest in answering this question comes from two sources. First, by analogy with Gorman's theory of exact aggregation, if there is cost heterogeneity, it will be natural to think of conditional input demands as dependent on c just as consumer demands depend on income or expenditure. The second reason is more involved. There is a fairly large literature which proposes solutions to the specification of *ex ante* cost functions when output is uncertain under potentially risk-averse behavior (e.g., Pope and Chavas 1994; Pope and Just 1998; Chambers and Quiggin 2000; Chavas 2008). The essential problem is that if inputs are applied *ex ante* under stochastic production, then the outputs in a) can't be observed. One approach is to make the assumptions required such that the *ex ante* cost function exists in an empirically convenient form. For example, given random supply shocks ε_i of the form

$$y_i = \bar{y}_i + H_i(\bar{\boldsymbol{y}}, \boldsymbol{z}, \varepsilon_i),\ E\left[H_i(\bar{\boldsymbol{y}}, \boldsymbol{z}, \varepsilon_i) \mid \boldsymbol{x}, \bar{\boldsymbol{y}}, \boldsymbol{z}\right] = 0,\ i = 1, \cdots, n_y, \tag{1}$$

the existence of a transformation function, $F(\boldsymbol{x}, \bar{\boldsymbol{y}}, \boldsymbol{z}) \leq 0$, defined over variable inputs, $\boldsymbol{x}$, planned outputs, $\bar{\boldsymbol{y}}$, and fixed inputs, $\boldsymbol{z}$, then the reasoning in Pope and Chavas (1994) implies the existence of a cost function in which $\bar{\boldsymbol{y}}$ replaces $\boldsymbol{y}$. That is, minimizing the variable cost of planned output yields $c = C(\boldsymbol{w}, \bar{\boldsymbol{y}}, \boldsymbol{z}) \equiv \min_{\boldsymbol{x}} \left\{ \boldsymbol{w}^{\mathsf{T}} \boldsymbol{x} : F(\boldsymbol{x}, \bar{\boldsymbol{y}}, \boldsymbol{z}) \leq 0, \boldsymbol{x} \geq \boldsymbol{0} \right\}$ for *all* von Neumann-Morgenstern utility functions in both static and dynamic environments. The conditional factor demands, $\boldsymbol{X}(\boldsymbol{w}, \bar{\boldsymbol{y}}, \boldsymbol{z})$, will continue to depend on the unobservable variables, $\bar{\boldsymbol{y}}$. However, these input demand functions only depend on $(\boldsymbol{w}, \boldsymbol{z}, c)$, all of which are observable, when a) reduces to b). Thus, the restrictions we seek are those that allow *ex ante* conditional demands to be studied using only observable variables.

To simplify notation by letting $\boldsymbol{y}$ now denote planned output, our main result on the dual restriction that is necessary and sufficient for $\boldsymbol{X}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z})$ to be written as $\tilde{\boldsymbol{X}}(\boldsymbol{w},c,\boldsymbol{z})$ is $c = C(\boldsymbol{w},\theta(\boldsymbol{y},\boldsymbol{z}),\boldsymbol{z}) \Leftrightarrow F(\boldsymbol{x},\theta(\boldsymbol{y},\boldsymbol{z}),\boldsymbol{z})$. That is, outputs must be weakly separable from variable inputs in the joint production technology, or equivalently, outputs must be weakly separable from variable input prices in the cost function. Though this result is somewhat restrictive in outputs,[1] it is fairly flexible in $\boldsymbol{x}$ and $\boldsymbol{z}$.

Duality and the Main Result

The neoclassical model of conditional demands for variable inputs with joint production, fixed inputs, and production uncertainty is

$$\boldsymbol{X}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) = \arg\min\left\{\boldsymbol{w}^{\top}\boldsymbol{x} : F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \le 0,\ \boldsymbol{x} \ge \boldsymbol{0}\right\}, \tag{2}$$

where $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}_{++}^{n_x}$ is an n_x–vector of variable inputs, $\boldsymbol{w} \in \mathcal{W} \subseteq \mathbb{R}_{++}^{n_x}$ is an n_x–vector of input prices, $\boldsymbol{y} \in \mathcal{Y} \subseteq \mathbb{R}_{++}^{n_y}$ is an n_y–vector of outputs, $\boldsymbol{z} \in \mathcal{Z} \subseteq \mathbb{R}_{++}^{n_z}$ is an n_z–vector of fixed inputs, $F : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R},\ F \in \mathcal{C}^{\infty},$ is the transformation function that defines the boundary of a closed, convex production possibilities set with free disposal in the inputs and the outputs, $\boldsymbol{X} : \mathcal{W} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X},\ \ \boldsymbol{X} \in \mathcal{C}^{\infty},$ is the n_x–vector of variable input demand functions, and $C(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) \equiv \boldsymbol{w}^{\top}\boldsymbol{x}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}),\ \ C : \mathcal{W} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}_{++},\ \ C \in \mathcal{C}^{\infty},$ is the variable

[1] Among other things it implies that marginal rates of product transformation are independent of the variable inputs and factor intensities.

cost function.[2] By Hotelling's/Shephard's Lemma, we have

$$\boldsymbol{X}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) = \nabla_w C(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) \equiv (\partial C / \partial w_1, \ldots, \partial C / w_{n_x})^{\intercal}, \tag{3}$$

where $^{\intercal}$ denotes vector/matrix transposition. Note that $\boldsymbol{X}$ is positively homogeneous of degree zero in $\boldsymbol{w}$. Integrating with respect to $\boldsymbol{w}$ to the variable cost function, we obtain

$$c = C(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{C}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})), \tag{4}$$

where $\theta : \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ is the *constant of integration*. In the present case, this means that θ *is constant with respect to* $\boldsymbol{w}$. The structure of θ cannot be identified from the variable input demands, and captures the structure of the joint production process relating to the fixed inputs and the outputs that is separable from the variable inputs.[3]

Under standard and well-known conditions, the variable cost function is strictly decreasing in $\boldsymbol{z}$, strictly increasing in $\boldsymbol{y}$, and convex in $(\boldsymbol{y},\boldsymbol{z})$. We are free to choose the *sign* of θ so that, without loss of generality $\partial\tilde{C}/\partial\theta > 0$.

Because $\tilde{C}$ is strictly increasing in θ, it has a unique inverse, $\theta = \gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},c)$, where $\gamma : \mathcal{W} \times \mathcal{Y} \times \mathcal{Z} \times \mathbb{R}_{++} \to \mathbb{R}$ is the inverse of $\tilde{C}$ with respect to θ. The function $\gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},c)$ is a *quasi-indirect production function*, analogous to the quasi-indirect utility function of

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[2] The paper focuses on interior solutions and smooth functions. The results can be extended in the standard way to corner solutions by a continuous extension of F or C to the boundary of the strictly positive orthant in $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ or $(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z})$ space (see, e.g., Blackorby, Primont, and Russell 1977). Also, smoothness can be relaxed to twice continuous differentiability with no change in the arguments that follow.

[3] We elucidate this point further in what follows.

consumer theory (Hausman 1981; Epstein 1982; LaFrance 1985, 1986, 1990, 2004; LaFrance and Hanemann 1989; von Haefen 2002). For all interior, feasible $(\boldsymbol{y},\boldsymbol{z}) \in \mathcal{Y} \times \mathcal{Z}$, γ is strictly increasing in *c*, strictly decreasing and quasi-convex in ***w***, and 0° homogeneous in (***w***,*c*). Two identities are simple implications of the inverse function theorem,

$$c \equiv \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, c)), \tag{5}$$

and

$$\theta \equiv \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta)). \tag{6}$$

This construction lets one write the conditional demands for the variable inputs as

$$\boldsymbol{x} = \nabla_{\boldsymbol{w}} \tilde{C} \equiv \boldsymbol{G}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, c). \tag{7}$$

One question of particular interest – answered below – is, "What is the necessary and sufficient condition for the conditional demands in a general model of joint production, (7), to reduce to $\boldsymbol{x} = \tilde{\boldsymbol{X}}(\boldsymbol{w}, c, \boldsymbol{z})$?"

Before addressing this question, we complete the development of the duality of variable cost functions in joint production. Define the *quasi-production function* by

$$\upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \equiv \min_{(\boldsymbol{w},c)} \left\{ \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, c) : \boldsymbol{w}^{\top} \boldsymbol{x} \leq c, \boldsymbol{w} \geq \boldsymbol{0}, c \geq 0 \right\}. \tag{8}$$

The name *quasi-production function* indicates that $\upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ only reveals that part of the structure of the joint production process associated with how the variable inputs interact directly with the fixed inputs and the outputs. As before, this is analogous to the situation where one only recovers that part of the structure of a direct utility function associated with the market demands for a subset of consumption goods.

From the identity $\theta(\boldsymbol{y}, \boldsymbol{z}) \equiv \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})))$, we have

$$
\begin{aligned}
\theta(\boldsymbol{y},\boldsymbol{z}) &\equiv \gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\tilde{C}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))) \\
&\geq \min_{(\boldsymbol{w},c)}\left\{\gamma(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},c) : \boldsymbol{w}^{\mathsf{T}}\boldsymbol{x} \leq c, \boldsymbol{w} \geq \mathbf{0}, c \geq 0\right\} \qquad (9) \\
&\equiv \upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}),
\end{aligned}
$$

for all interior feasible $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$. The inequality follows from the fact that $\theta(\boldsymbol{y},\boldsymbol{z})$ is feasible but is not necessarily optimal in the minimization problem. Because $F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = 0$ defines the boundary of the production possibility set, and since F is strictly increasing in $\boldsymbol{y}$ and strictly decreasing in $\boldsymbol{z}$, $\theta(\boldsymbol{y},\boldsymbol{z}) = \upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ is logically equivalent to $F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = 0$. That is, the quasi-production function is defined equivalently as the unique solution with respect to θ of the implicit function,[4]

$$
F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})) = 0. \qquad (10)
$$

What class of variable cost functions generates conditional input demand equations in the form,

$$
\boldsymbol{x} = \tilde{\boldsymbol{X}}(\boldsymbol{w},c,\boldsymbol{z})? \qquad (11)
$$

We can now prove the following:

Proposition*: The variable input demand equations have the structure* (11) *if and only if the variable cost function has the weakly separable structure*

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[4] The existence of $\theta : \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ is not an issue here. For example, one could *always* define the function $\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})) \equiv F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) + \theta(\boldsymbol{y},\boldsymbol{z})$, with $\theta(\boldsymbol{y},\boldsymbol{z}) \equiv 0$, and all of the properties listed above are met. The issue is when θ is the *only* way that $\boldsymbol{y}$ enters the joint production technology.

$$c = C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) \equiv \tilde{C}(\boldsymbol{w}, \theta(\boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z}), \tag{12}$$

and the variable cost function has the weakly separable structure (12) *if and only if the joint production transformation function has the weakly separable structure,*

$$F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \equiv \tilde{F}(\boldsymbol{x}, \theta(\boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z}). \tag{13}$$

Proof: First, differentiating (12) with respect to $\boldsymbol{w}$, Shephard's Lemma implies,

$$\boldsymbol{x} = \nabla_{\boldsymbol{w}} \tilde{C}. \tag{14}$$

$\tilde{C}$ is strictly monotonic in and has a unique inverse with respect to θ, say $\theta = \tilde{\gamma}(\boldsymbol{w}, \boldsymbol{z}, c)$. Substituting this into (14) obtains

$$\boldsymbol{x} = \nabla_{\boldsymbol{w}} \tilde{C}(\boldsymbol{w}, \tilde{\gamma}(\boldsymbol{w}, \boldsymbol{z}, c), \boldsymbol{z}) \equiv \tilde{\boldsymbol{X}}(\boldsymbol{w}, c, \boldsymbol{z}). \tag{15}$$

Second, and conversely, integrating (15) with respect to $\boldsymbol{w}$ returns a variable cost function with the separable structure in (12), where $\theta(\boldsymbol{y}, \boldsymbol{z})$ is again the constant of integration for the system of partial differential equations.

Third, if the representation of technology has the separable structure in (13), it follows that

$$\arg\min\left\{\boldsymbol{w}^{\top}\boldsymbol{x} : \tilde{F}(\boldsymbol{x}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) \leq 0,\ \boldsymbol{x} \geq \boldsymbol{0}\right\} \equiv \tilde{\boldsymbol{X}}(\boldsymbol{w}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})). \tag{16}$$

This implies that the variable cost function has the separable structure

$$\boldsymbol{w}^{\top}\tilde{\boldsymbol{X}}(\boldsymbol{w}, \theta(\boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z}) \equiv \tilde{C}(\boldsymbol{w}, \theta(\boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z}). \tag{17}$$

So far we have shown that $\tilde{F}(\boldsymbol{x}, \theta(\boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z}) \leq 0 \ \Rightarrow \tilde{\boldsymbol{X}}(\boldsymbol{w}, \theta(\boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z}) \ \Leftrightarrow c = \tilde{C}(\boldsymbol{w}, \theta(\boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z})$. To show that $c = \tilde{C}(\boldsymbol{w}, \theta(\boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z}) \Rightarrow \boldsymbol{x} = \tilde{\boldsymbol{X}}(\boldsymbol{w}, \theta(\boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z})$, one can proceed in one of two ways. The way first is to note that, given monotonicity of $\tilde{C}$ in θ and the smoothness assumption, one can apply the results of Primont and Sawyer (1993) to recover the technically efficient representation of technology.

Perhaps a more direct and illuminating approach to establish that (12) implies the weakly separable production technology in the Proposition is to use the concept of the quasi-production function which satisfies

$$\tilde{\upsilon}(\boldsymbol{x},\boldsymbol{z}) \equiv \min_{(\boldsymbol{w},c)} \left\{ \tilde{\gamma}(\boldsymbol{w},\boldsymbol{z},c) : \boldsymbol{w}^{\mathsf{T}}\boldsymbol{x} \le c, \boldsymbol{w} \ge \boldsymbol{0}, c \ge 0 \right\}. \tag{18}$$

By the same logic that leads to (9) above, $\theta(\boldsymbol{y},\boldsymbol{z}) \equiv \gamma(\boldsymbol{x},\boldsymbol{z},\tilde{C}(\boldsymbol{x},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))) \ge \tilde{\upsilon}(\boldsymbol{x},\boldsymbol{z})$ for all interior feasible $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$, with the boundary of the closed, convex feasible production possibilities set defined by equality on the far right. The marginal rates of transformation between outputs are therefore independent of the variable inputs,

$$\frac{\partial}{\partial x_k}\left(\frac{\partial \theta(\boldsymbol{y},\boldsymbol{z}) / \partial y_i}{\partial \theta(\boldsymbol{y},\boldsymbol{z}) / \partial y_j} \right) = 0, \ \forall\ i, j, k. \tag{19}$$

Hence, $\boldsymbol{y}$ is weakly separable from $\boldsymbol{x}$ in the transformation function (Goldman and Uzawa 1964, Lemma 1). Therefore, since $F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = 0$ defines the boundary of the production possibility set and F is strictly increasing in $\boldsymbol{y}$, it follows that $\tilde{\upsilon}(\boldsymbol{x},\boldsymbol{z}) = \theta(\boldsymbol{y},\boldsymbol{z})$ is equivalent to $\tilde{F}(\boldsymbol{x},\theta(\boldsymbol{y},\boldsymbol{z}),\boldsymbol{z}) = 0$. ■

Conclusions

An empirically important question concerns when cost-minimizing input demands can be stated in terms of empirically observable *ex ante* data: costs, input prices, and fixed or quasi-fixed inputs. We conclude that separability of expected output from variable inputs must occur in technology and similarly separability of expected or planned outputs from input prices must occur in the cost function. If these restrictions are deemed too strong, then alternative approaches to cost function formulation must be pursued.

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