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Repeated Auctions with the Right of First Refusal

By

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Repeated Auctions with the Right of First Refusal

Abstract: This paper characterizes a set of Nash equilibria in a first-price sealed-bid repeated auction with the right of first refusal using two bidders and asymmetric information regarding the bidders' value distributions. When contract value is constant from one auction to the next and winners' values are publicized, agents retain the value of incumbency and bids are identical to one-shot auctions. When each agents' contract values are random across auctions, agents choose to bid away the full expected value of incumbency, providing a measure of the value of information in this context.

1 Introduction

The right of first refusal exists in many types of contracts that can be thought of as repeated auctions. Undeveloped land, residential property, and commercial property contracts all may include the right of first refusal to provide current tenants the option to retain the contract to the property. Securities sales by owners often allow other owners the right of first refusal in order to control ownership. Employment contracts, especially those of athletes and entertainers may empower the current employer with the right of first refusal as encouragement to support those unproven talents early in their careers. Various types of investment settings provide investors the right of first refusal on the developed innovation as an incentive to invest in research. Often, procurement contracts such as those for municipal garbage collection or cable television service, include the right of first refusal. Some National Parks concession contracts also include the right of first refusal.

A repeated auction is defined in this article to be an infinite series of auctions used to award the same, indivisible object in each period. Some repeated auctions incorporate the right of first refusal (rofr), where the incumbent has the opportunity to continue to hold the right to the awarded object by matching the best bid of any entering bidder in each period. We provide a general analysis of initial period, entrant and incumbent bidder behavior. Additionally, we derive the equilibrium bids when contract values are constant over time and made common knowledge, and in the case where values are stochastic across auctions.

Repeated auctions have been studied since Demsetz (1968) called for the use of franchise bidding to award monopoly contracts. The treatment of and assumptions concerning the incumbent affects the outcome of the repeated auction. The importance of the incumbent is seen in Osmundsen (1996) when repeated auctions are applied to nonrenewable resource extraction franchises, and in Laffont and Tirole (1998) as incumbent's make investment choices in the first period. Here, the specifics of the incumbent may represent actual auctions, and allow the characterization of an equilibrium.

Repeated auctions are often assumed to include the same bidders in each period, and numerous articles therefore examine collusion among repeat bidders as an important aspect of the repeated auction process (Phillips et al. 2003, Fabra 2003, Aoyagi 2003, Skrzypacz and Hopenhayn 2004). However, the length of the awarded contracts or the characteristics of the object make the assumption of a changing pool of entrant bidders more realistic. In this work, the entrant bidders are therefore allowed to vary across periods.

This article is the first to model the equilibria in first price repeated auctions with the right of first refusal. Others have considered the right of first refusal in other contexts. Walker (1999) argues that the right of first refusal protects against bargaining breakdown and inhibits exit from a market, but it also may limit competitive bidding. In the first economic analysis of the right of first refusal, Kahan (1999) models the value of the right of first refusal in negotiated contracts (not auctions), and shows that the value of the right will depend on the relative valuations of the good. Additional theory and an experimental examination of the right of first refusal is presented in Grosskopf and Roth (2004), who conclude that the specific characteristics of the right of first refusal can work to the advantage or disadvantage of the right holder. Bikhchandani et al. (2005) discuss the impacts of the right of first refusal on the seller and potential buyers, and conclude that the right is inefficient (the bidder with the highest value does not necessarily win), and the seller may forego surplus relative to auctions without the right of first refusal. Finally, Chouinard (2005) compares one-shot first-price auctions with and without the right of first refusal in the context of U.S. National Parks concession contracts.

Building on Bikhchandani et al. (2005) and Chouinard (2005), equilibria of first price repeated auctions with the right of first refusal are modelled. Two bidders per round are considered, each with uniform value functions. Initially, it is assumed that the contract value is constant over time for each agent, and the winners' values are publicly revealed. The model is then extended to consider a case in which private contract value is stochastic across agents and across time for each agent, such that the incumbent's assessment of the value of the next contract is known publicly, and the value of future contracts are not known with certainty by any agent.

These informational differences lead to different bidding strategies for both the initial auction and all subsequent auctions. With constant contract values for each agent, the first-round bid is a standard first-price one-shot auction strategy: agents will bid one half of their contract valuation, such that the winner retains half of the current contract value plus the expected value of incumbency, and the agent with the highest valuation wins. With temporally stochastic contract value, first-round agents bid one half the current contract value plus the entire expected value of incumbency. After the initial round, entrants with temporally constant contract value will bid only if their contract value is larger than the known value of the incumbent. With temporally stochastic contract value, an entrant will bid one half their current contract value plus at least half of the value of incumbency in an attempt to beat the right of first refusal held by the incumbent.

2 General ROFR auction environment

In each period there are two bidders, and although the number of agents is restricted to two per round, agents may hold different contract valuations in different rounds.[1] At the beginning of each period, the seller offers for bid a monopoly right for one period. The largest bid wins the monopoly right for that period. The winner transfers her bid amount to the seller, and retains the value of the monopoly right for that period.

In the first auction, agents submit bids simultaneously.[2] This initial round amounts to a symmetric private value auction. In each subsequent round, the entrant submits a bid first. The incumbent learns the entrant's bid, and then matches the entrant's bid if and only if the incumbent's expected value of winning exceeds the entrant's bid. If the incumbent fails to submit a matching bid to secure the contract, the entrant receives the contract and the incumbency until she is outbid by a new entrant in some future round. We assume for simplicity that an ex-incumbent never again enters the auction series after losing an incumbency.

Contract value for each agent is independently drawn from a standard uniform distribution, and this fact is common knowledge among agents. The actual value of a contract in any given period is private knowledge until and unless this value is revealed through the auction process, and no collusion of any form exists among

[1] This can be interpreted in two ways: bidders may change from period to period, or bidders draw new valuations in each period. More on this later.

[2] If two bidders submit identical bids in the initial auction, a winner will be selected randomly.

bidders. The payoff function can be characterized as the present value of the initial contract plus the expected value of future contracts as an incumbent, for agent i in period t, can be written as

$$\mathrm{E}[U_t^i|\mathrm{win}] = \begin{cases} (v_t^i - b_t^i) + \delta \mathrm{E}[U_{t+1}^I] & \text{if } b_t^i > b_t^j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

where $\mathbf{b}_t = \begin{bmatrix} b_t^1 & b_t^2 \end{bmatrix}$ and $\mathbf{v}_t = \begin{bmatrix} v_t^1 & v_t^2 \end{bmatrix}$ are bids and values for agents 1 and 2 in time t. $\mathrm{E}[U_{t+1}^I]$ is the expected value of being the incumbent beginning in period $t+1$, and δ is the discount factor such that $\delta\mathrm{E}[U_{t+1}^I]$ is the expected present value of incumbency.

Incumbents have the right of first refusal and therefore bid in response to an entrant's bid. This bid pattern means that the value of incumbency depends crucially on entrant bids. Below we characterize the incumbent's problem, the entrant's problem, and the initial-period problem in general terms. In the next section we develop optimal bid functions for more specific assumptions about how contract value changes over time.

2.1 Incumbent's bid in response to an entrant.

The incumbent's expected net present value of the auction series evaluated at time t, with privately known v_t^I and entrant bid b_t^E is the sum of this period's net earnings plus the discounted expected present value of incumbency. Applying Bayes' theorem, this value can be written as

$$\begin{aligned}
\mathrm{E}[U_t^I] &= (v_t^I - b_t^E) + \delta \mathrm{E}[U_{t+1}^I | U_{t+1}^I > b_{t+1}^E] \Pr[U_{t+1}^I > b_{t+1}^E] \\
&= (v_t^I - b_t^E) + \delta \mathrm{E}[U_{t+1}^I] \qquad (2)
\end{aligned}$$

where δ is the discount rate, $\mathrm{E}[U_{t+1}^I]$ is the expected value of incumbency for t+1 onward, and v_t^I is known when the entrant's bid b_t^E is offered. Given an entrant's bid, the incumbent's optimal bid is

$$b_t^I = \begin{cases} b_t^E & \text{if } \mathrm{E}[U_t^I] \geq b_t^E \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

The optimal bid of an entrant facing a current incumbent and the expected value of future incumbency will be derived next.

2.2 Entrant's bid against an incumbent.

The expected value to the entrant of bidding against an incumbent is the net value of the first period's earnings plus the discounted present value of the incumbency that the entrant might win, times the probability of winning the incumbency. An entrant challenging an incumbent solves

$$\max_{b_t^E} \ \mathrm{E}[U_t^E] = \big((v_t^E - b_t^E) + \delta \mathrm{E}[U_{t+1}^I]\big) \Pr[\mathrm{E}[U_t^I] < b_t^E], \qquad (4)$$

where the entrant's bid b_t^E is now a choice variable. Relying on Bayes' theorem,

$$
\begin{aligned}
\mathrm{E}[U_{t+1}^I] = \big(&\mathrm{E}[v_{t+1}^I - b_{t+1}^E | U_{t+1}^I > b_{t+1}^E] + \\
&\delta \mathrm{E}[U_{t+2}^I | U_{t+2}^I > b_{t+2}^E] \Pr[U_{t+2}^I > b_{t+2}^E]\big) \Pr[U_{t+1}^I > b_{t+1}^E] \\
= &\mathrm{E}[v_{t+1}^I - b_{t+1}^E] + \delta \mathrm{E}[U_{t+2}^I] \Pr[U_{t+1}^I > b_{t+1}^E]. \qquad (5)
\end{aligned}
$$

Information about future periods is constant across all future periods such that $\mathrm{E}[U_{t+1}^I] = \mathrm{E}[U_{t+2}^I]$, so equation 5 can be rearranged as

$$
\mathrm{E}[U_{t+1}^I] = \frac{\mathrm{E}[v_{t+1}^I - b_{t+1}^E]}{1 - \delta \Pr[U_{t+1}^I > b_{t+1}^E]}. \qquad (6)
$$

the numerator $\mathrm{E}[v_{t+1}^I - b_{t+1}^E]$ depends in part on assumptions about the characteristics of future contract values v_t^i, and $\Pr[U_{t+1}^I > b_{t+1}^E]$ is the probability that an incumbent renews a contract. Given equation 5, this probability can be written more specifically as

$$
\begin{aligned}
\Pr[U_{t+1}^I > b_{t+1}^E] &= \Pr[v_{t+1}^I - b_{t+1}^E + \delta \mathrm{E}[U_{t+2}^I] > b_{t+1}^E] \\
&= 1 - \Pr[v_{t+1}^I < 2b_{t+1}^E - \delta \mathrm{E}[U_{t+2}^I]]. \qquad (7)
\end{aligned}
$$

Because $\mathrm{E}[U_{t+1}^I] = \mathrm{E}[U_{t+2}^I]$, the right-hand-side of equation 6 can be substituted into equation 7 and solved for $\Pr[U_{t+1}^I > b_{t+1}^E]$.

To more completely characterize equations 4 through 7 and to derive the optimal bid function, we must impose a precise specification of the stochastic nature of v_t^i. This will be done in section 3.

2.3 First round bidding.

Bidder i's expected payoff of winning the initial auction is similar to an entrant trying to outbid an incumbent, but in this case each agent is on equal footing. Each agent bids without knowledge of the other agent's first period bid or contract valuation. As before, the probability of an agent winning is equal to the probability that an agent's bid is higher than the other agent's bid. Because agents are identical, we can infer that their bid functions will be symmetric, and an agent's bid need not be larger than the other agent's valuation of a win (as is the case with an entrant challenging an incumbent). The first-period maximization problem for each agent is

$$\max_{b_0^i} \mathrm{E}[U_0^i] = \big((v_0^i - b_0^i) + \delta \mathrm{E}[U_1^I]\big) \Pr[b_0^j < b_0^i], \quad i \neq j. \tag{8}$$

The following discussion will focus on agent 1's perspective, so that b_0^1 is the choice variable, and b_0^2 is unknown from agent 1's perspective. Let $v_t^i = v_0^1$ be the value draw for agent $i = 1$ at $t = 0$, and b_0^i and b_0^j are agent $i = 1$ and agent $j = 2$ first round bids, respectively. The probability of agent 1 winning is $\Pr[b_0^2 < b_0^1]$.

To reduce notational clutter, let $\tilde{v}_0^i = v_0^i + \delta \mathrm{E}[U_1^i]$, such that agent i's optimal bid is a function of $\tilde{v}_0^i$: $\tilde{b}_0^i = \tilde{b}_0^i(\tilde{v}_0^i)$. The probability of agent 1 winning is then $\Pr[\tilde{b}_0^2(\tilde{v}_0^2) < b_0^1]$. Now define $\tilde{\sigma}(\tilde{b}_0^i)$ as the inverse function of $\tilde{b}^i(\tilde{v}_0^i)$ such that $\tilde{\sigma}(\tilde{b}_0^i(\tilde{v}_0^i)) = \tilde{v}_0^i$. It follows that

$$\Pr[\tilde{b}_0^2(\tilde{v}_0^2) < \tilde{b}_0^1] = \Pr[\tilde{v}_0^2 < \tilde{\sigma}(\tilde{b}_0^1)] = \Pr[v_0^2 < \tilde{\sigma}(\tilde{b}_0^1) - \mathrm{E}[U_2^I]], \tag{9}$$

so we can restate equation 8 for agent 1 as

$$\begin{aligned}\max_{b_0^1}\ \mathrm{E}[U_0^1] &= (\tilde{v}_0^1 - b_0^1)\Pr[\tilde{v}_0^2 < \tilde{\sigma}(b_0^1)] \\ &= ((v_0^1 - b_0^1) + \mathrm{E}[U_1^I])\Pr[v_0^2 < (\tilde{\sigma}(b_0^1) - \mathrm{E}[U_2^I])]. \end{aligned} \tag{10}$$

The optimal bid functions for the entrant based on equation 4 and the optimal bid function for first-period bidders based on equation 10 depend on the stochastic characteristics of v_t^i. The next section builds upon this foundation to identify optimal bid functions under two different informational scenarios.

3 Special cases

The two scenarios examined below differ only in how contract values for a given agent change over time. Assume for both cases that contract value v_t^i is drawn independently across across agents from a standard uniform distribution, such that the PDF and CDF of v_t^i are $f(v_t^i) = 1$ and $F(v_t^i) = v_t^i$, respectively. This is a common distributional assumption in the auction literature.

In the first scenario, each agent draws a contract value prior to the initial period and retains this value for all subsequent auctions. In the second scenario, each agent draws a new contract value prior to bidding on each successive auction. This change in assumption leads to substantially different bidding strategies and buyer/seller welfare distribution.

3.1 Contract value constant over time

In this scenario, each agent independently draws $v_t^i = v^i$ prior to the first round auction from a standard uniform distribution (with support [0,1]). This value is constant over time for each agent, although incumbents need not face the same challenger (entrant) in subsequent periods. The value of the winner is made common knowledge. This is similar to Vickrey (1961), where it is assumed one bidder knows the valuation of the other. Landsberger et al. (2001) evaluate first price auctions given the ranking of bidder valuations is known. They suggest that knowledge about incumbent's values may also come from previous auction experience, access to other's financial resources, or other idiosyncratic features of bidders. In the types of auctions considered here, it is reasonable that entrant bidders may be able to observe the incumbent, as one period of the repeated auction may last several years. This may allow entrants to discover the incumbent's contract value even if winning bid values are hidden.

One practical implication of this information structure is that an entrant bidding against an incumbent will not submit a meaningful bid unless she will win the contract. Without further contractual rules, having no challenging bid would allow the incumbent to renew her contract with some arbitrarily small bid and extract virtually all rents from the seller. It is therefore assumed that the seller will not accept a bid from an incumbent lower than the bid with which she first won the contract. In this setting, entrant and incumbent bids are relatively straightforward, and will be formally characterized after deriving the bid functions for the initial auction.

3.1.1 Initial period bids

To maximize the objective function characterized by equation 10 for the current environment, the probability of i outbidding j ($\Pr[b_0^j < b_0^i]$) and the discounted expected value of being the incumbent ($\mathrm{E}[U_t^I]$) must be determined.

First consider the expected value of incumbency $\mathrm{E}[U_1^I]$ as characterized in general form in equation 5. In this environment, an incumbent's bids will be constant for the duration of her tenure because a) an entrant will bid only if she will win, and b) incumbents must bid at least their initial bid but have no incentive to bid more than their initial bid. It follows that for each period, an incumbent will receive $v^I - b^I$.

The probability of an incumbent winning one (additional) auction is the probability that her expected value of incumbency is greater than that of a challenging entrant. Because contract value and an incumbent's bids are constant over time, the incumbent's value of future incumbency is at least as large as an entrant's value of future incumbency iff $v^E \leq v^I$. Given known v^I and a standard uniform distribution for v^E, the probability that an incumbent wins in any given period is $\Pr[v^E < v^I] = v^I$. Under these conditions, the expected value of incumbency (equations 5 and 6) can be written as

$$\begin{aligned} \mathrm{E}[U_1^I] &= \left(\mathrm{E}[v_1^I - b_1^E] + \delta \mathrm{E}[U_2^I]\right) \Pr[U_1^I > b_1^E] \\ &= \left((v^I - b^E) + \delta \mathrm{E}[U_1^I]\right) v^I \\ &= \frac{(v^I - b^E)v^I}{1 - \delta v^I}. \end{aligned} \tag{11}$$

Substituting the right-hand-side of equation 11 into the initial bidder's problem (equation 10) and proceeding from bidder 1's perspective, bidder 1 solves

$$\begin{aligned} \max_{b^1} \mathrm{E}[U_0^1] &= (\tilde{v}^1 - b^1) Pr(b^2 < b^1) \\ &= \left[(v^1 - b^1) + (v^1 - b^1)\left(\frac{\delta v^1}{1 - v^1\delta}\right)\right] Pr(b^2(\tilde{v}^2) < b^1) \\ &= \left(\frac{v^1 - b^1}{1 - v^1\delta}\right) \Pr[\tilde{v}^2 < \tilde{\sigma}(b^1)]. \end{aligned} \tag{12}$$

In this case, $Pr(\tilde{v}^2 < \tilde{\sigma}(b^1))$ can be simplified. Contract values are constant across time and $\tilde{v}^1 > \tilde{v}^2$ iff $v^1 > v^2$, so $\Pr[\tilde{v}^2 > \tilde{v}^1] = \Pr[v^2 > v^1]$. We can therefore replace

$\Pr[\tilde{v}^2 < \tilde{\sigma}(b^1)]$ with $\Pr[v^2 < \sigma(b^1)]$, where $\sigma(b^1)$ is an inverse function analogous to $\tilde{\sigma}(b^1)$. Given that v^2 is uniformly distributed,

$$\begin{aligned} Pr(b_2(v_2) < b^1) &= \Pr[v_2 < \sigma(b^1)] \\ &= F(\sigma(b^1)) \\ &= \sigma(b^1). \end{aligned} \tag{13}$$

Further, note that because the expected value of incumbency is a function of v^I, the present value of winning the first round collapses into $\left(\frac{v^1-b^1}{1-v^1\delta}\right)$. Making these substitutions leads to

$$\max_{b^1} \mathrm{E}[U_0^1] = \left(\frac{v^1 - b^1}{1 - v^1\delta}\right)\sigma(b^1). \tag{14}$$

The first order condition associated with this maximization is

$$\sigma'(b^1) = \frac{\sigma(b^1)}{v^1 - b^1}. \tag{15}$$

Because $\sigma(b^1(v^1)) = v^1$, replace v^1 in first-order-condition 15 with $\sigma(b^1)$:

$$\sigma'(b^1) = \frac{\sigma(b^1)}{\sigma(b^1) - b^1}.$$

Solving this differential equation yields $\sigma(b^1) = 2b^1$ and $\sigma'(b^1) = 2$.[3] Because $\sigma(b^1) = v^1$, the optimal strategy for bidder 1 is

$$b^1 = \frac{v^1}{2}. \tag{16}$$

[3]To solve this differential equation, let $y(b^1) = \sigma(b^1) - b^1$, so that $\sigma(b^1) = y(b^1) + b^1$ and $\sigma'(b^1) = y'(b^1) + 1$. Substituting for σ and σ' yields $y'(b^1) * y(b^1) = b^1$. The solution to this equation using $y(0) = 0$ and integration by parts is $\frac{y(b^1)^2}{2} = \frac{(b^1)^2}{2}$. Only using the positive solution gives $y(b^1) = b^1$. Given $\sigma(b^1) = y(b^1) + b^1$, substituting for $y(b^1)$ yields $\sigma(b^1) = 2b^1$.

In this initial auction the two bidders are symmetric, so each bidder i will bid $b_i = \frac{v_i}{2}$ and the highest value will win the auction in the first period.[4] Because v^1 is the gross value of the first-period contract, the result indicates that the winner retains one half of the first-period contract value and the entire present value of incumbency. Interestingly, this result is identical to a one-shot first-price auction.

3.1.2 Entrant and incumbent bidding strategies

For subsequent rounds given this environment, the entrant and incumbent bids are relatively straightforward. From the entrant's perspective, $\Pr[\mathrm{E}[U_t^I] < b_t^E]$ in equation 4 is either zero or one. Equation 12 implies that the value of winning an auction (conditional on winning it) is $\frac{v^i - b^i}{1-\delta v^i}$. This value is nonpositive if $b^i \geq v^i$. An incumbent will match an entrant's bid only if $v^I \geq b^E$, and an entrant will therefore bid only if $v^E > v^I + \varepsilon$. The optimal entrant bid is therefore

$$b^E = \begin{cases} \emptyset & \text{if } v^E \leq \ v^I \\ v^I + \varepsilon & \text{if } v^E > v^I. \end{cases}$$

If no entrant bids, the incumbent submits her previous bid as required by the seller, but an entrant will bid only if she knows she can win. Therefore, the incumbent's initial bid of $b^i = \frac{v^i}{2}$ will stand for each period until an entrant with a higher contract value enters the auction.

[4]Chouinard (2006) considers two generalizations of this case: first, N bidders are allowed, and second, the cumulative distribution function for v_i is generalized to $F = v^a$ for $v \in (0, 1)$. These generalization complicates the analytics somewhat, but the results are similar to the ones presented in this article. or example, for N bidders, the optimal initial bid is $b_i = v_i(N-1)/N$. For the generalized distribution, the optimal bid is $b_i = v_i a/(a+1)$.

3.2 Uncertain future contract value

Now, we allow contract value to differ over time for each agent. At the beginning of each period, individuals privately draw v_t^i from independent and identical standard uniform distributions, and then submit a bid.[5] Thus, each agent knows their own current contract value at bidding time, but have symmetric information about future period values. A fundamental difference in this scenario is that the value of future incumbency is independent of any agent's current bid.

The expected value of being an incumbent is first derived below, followed by the objective function and the bid function of an entrant, which is more complicated than in the previous scenario. The bid function for initial bidders is derived last.

3.2.1 The value of incumbency and the entrant's bid

The general form of the entrant's problem (equation 4) depends on the expected value of incumbency (equation 6). The expected value of incumbency is based on the expected value $\mathrm{E}[v^I]$ and the entrant's optimal bid, for future periods, and the probability of retaining the incumbency in future periods. Given this setting, $\mathrm{E}[v_{t+1}^I - b_{t+1}^E] = \mathrm{E}[v^I] - b^E(\mathrm{E}[\mathbf{v}])$, and with standard uniform distributions for all v_t^i, $\mathrm{E}[\mathbf{v}] = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$ for all t. Because expectations about future periods are constant over time, the time index and expectation operator in equation 6 will be dropped and replaced with an overbar, so $\bar{U}^I \equiv \mathrm{E}[U_{t+i}^I]$ represent the expected value of incumbency (given incumbency), and $\bar{P}^I \equiv \Pr[\bar{U}^I > \bar{b}^E]$ represents the probability of an incumbent winning again in future periods. Remembering that $\bar{b}^E$ is a choice variable evaluated

[5] These can be thought of as predictions for period t contract value upon which agents base their bids.

at expected values, it follows from equation 6 that

$$\mathrm{E}\left[\bar{U}^I\right] = \frac{0.5 - \bar{b}^E}{1 - \delta \bar{P}^I}. \tag{17}$$

It also follows from equation 7 that the probability of the incumbent winning in future periods is [6]

$$\begin{aligned} \bar{P}^I &= 1 - (2\bar{b}^E - \delta \bar{U}^I \bar{P}^I) \\ &= \frac{1 - 2\bar{b}^E}{1 - \delta \bar{U}^I} \end{aligned} \tag{18}$$

Substituting equation 18 into equation 17 and solving for $\bar{U}^I$ provides[7]

$$\bar{U}^I(\bar{b}^E) = \frac{1 + \delta(\bar{b}^E - 0.5) - \sqrt{(\delta(\bar{b}^E - 0.5)^2) + 4\delta(\bar{b}^E - 0.5)}}{2\delta}. \tag{19}$$

Substituting $\bar{U}^I(\bar{b}^E)$ from 19 into the entrant's current-period objective function (equation 4) provides

$$\mathrm{E}[U_t^E] = \left((v_t^E - b_t^E) + \delta \bar{U}^I(\bar{b}^E)\right) \Pr[U_t^I < b_t^E], \tag{20}$$

where v_t^E is known by the entrant at bidding time, and

$$\begin{aligned} \Pr[U_t^I < b_t^E] &= \Pr[v_t^I < 2b_t^E - \delta \bar{U}^I \bar{P}^I] \\ &= 2b_t^E - \delta \bar{U}^I \bar{P}^I, \end{aligned} \tag{21}$$

[6]The probability in equation 18 can be written as an infinite sum of present expected values: $\bar{P}^I = (1 - 2\bar{b}^E) \sum_{i=0}^{\infty} (\delta \bar{U}^I)^i$. Also, $\Pr[\bar{U}^I < \bar{b}^E] = \frac{2\bar{b}^E - \delta \bar{U}^I}{1 - \delta \bar{U}^I}$.

[7]There are actually two solutions, because equation 17 is quadratic in $\bar{U}^I$ after the substitution of equation 18. The only economically meaningful solution is that which is a declining function of $\bar{b}^E$, because the expected value of incumbency must decline as the expected entrant bid increases.

with $\bar{U}^I$ and $\bar{P}^I$ defined by equations 19 and 18 above. Notice that equation 20 is a function of two potentially different entrant bids: an entrant bid for period t and the expected entrant bid for future periods. To solve for these entrant bids, we maximize equation 20 twice: once to derive the expected future bid $\bar{b}^E$, and a second time to derive b_t^E conditional on $\bar{b}^E$. To derive $\bar{b}^E$, we maximize equation 20 conditional on $v_t^E = \mathrm{E}[v_t^E] = 0.5$ and the restriction that $b_t^E = \bar{b}^E$. After all the relevant substitutions from above, equation 20 is a complicated function of parameters δ, $\mu = 0.5$, and the choice variable $\bar{b}^E$. Maximizing this function provides the optimal entrant bid $\bar{b}^E$.[8]

The optimal $\bar{b}^E$ is then substituted into equations 18 and 19, which allows calculation of a scalar value for the expected present value of incumbency, $\delta\bar{U}^I\bar{P}^I$. This scalar value is substituted into the entrant's objective function (20), which is maximized (again) for the optimal period t bid, b_t^E. With a scalar value for $\delta\bar{U}^I\bar{P}^I$ in hand, equation 20 is quadratic in b_t^E. The first-order condition is

$$\frac{\partial \mathrm{E}[U_t^E|\bar{U}^I, \bar{P}^I]}{\partial b_t^E} = 2(v_t^E - b_t^E + \delta\bar{U}^I) - (2b_t^E - \delta\bar{U}^I\bar{P}^I) = 0.$$

Solving for b_t^E provides the optimal entrant bid

$$b_t^E(v_t^E) = \frac{1}{2}\left(v_t^E + \delta\bar{U}^I\left(1 + \frac{\bar{P}^I}{2}\right)\right). \tag{22}$$

Equation 22 shows that the entrant will bid away one half of the current period's contract value plus a fraction (between one-half and three-quarters) of the expected value of incumbency, depending on $\bar{P}^I$. Figure 1 shows the expected value of incumbency as a function of expected entrant bids. As the expected entrant bid increases, the expected value of incumbency declines. Notice also that there is no analogous

[8] *Mathematica*©code and results are available from the authors upon request.

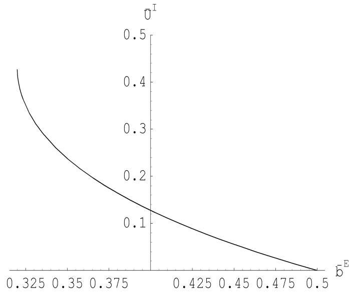

Figure 1: Expected value of future incumbency (eq. 19).

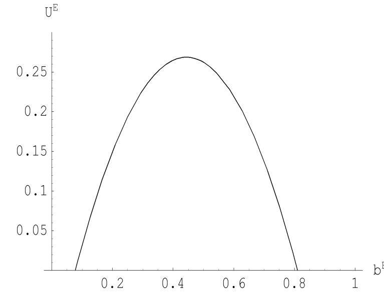

Figure 2: Entrant contract value as a function of b_t^E (eq. 22)

continuous entrant bid function for the case in which agents have static contract values: in that case, entrants either bid if the value of their incumbency outweighs the incumbent's value, or they do not bid at all.

Figure 2 shows the expected value of a contract from the entrant's perspective (with known v_t^E), as a function of the entrant's bid. Both figures are based on $\delta = 1/1.05$ and $v_t^E = \mathrm{E}[v_t^E] = 0.5$. Given these values, the expected entrant bid in future periods is 0.328; the discounted expected value of incumbency ($\delta \bar{U}^I$) is 0.31; $\bar{P}^I$ is 0.497; the optimal entrant bid is 0.44, and the expected value of an entrant's optimal bid given current contract value of $v_t^E = 0.5$ is 0.269.

3.2.2 First-round bidding

Given that $\mathrm{E}[U_{t+i}^I] = \bar{U}^I$ for all $i > 0$ and the uniform distribution of v_t^i, the objective function for an initial bidder (equation 8) becomes

$$\begin{aligned}\max_{b_0^1}\ \mathrm{E}[U_0^1] &= ((v_0^1 - b_0^1) + \delta \bar{U}^I)\Pr[v_0^2 < (\tilde{\sigma}(b_0^1) - \delta \bar{U}^I)] \\ &= ((v_0^1 - b_0^1) + \delta \bar{U}^I)(\tilde{\sigma}(b_0^1) - \delta \bar{U}^I). \qquad (23)\end{aligned}$$

The first-order condition for this problem is the differential equation

$$\tilde{\sigma}'(b_0^1) = \frac{\tilde{\sigma}(b_0^1) - \delta\bar{U}^I}{\tilde{v}_0^1 - b_0^1}, \tag{24}$$

As a reminder, $\tilde{v}_0^i = v_0^i + \delta\bar{U}^I$, so this first order condition is different than the analogous equation 15 by the presence of the constant $\delta\bar{U}^I$ in two places. A non-empty solution to this differential equation is

$$\tilde{\sigma}(b_0^1) = 2b_0^1 - 3\delta\bar{U}^I.^9 \tag{25}$$

Finally, because of symmetry among agents, $\Pr[v_0^2 + \delta\bar{U}^I) < (\tilde{\sigma}(b_0^1)] = \Pr[v_0^2 < v_0^1]$, implying that $\tilde{\sigma}(b_0^1) = v_0^1 + \delta\bar{U}^I$. Substituting the right hand side of equation 25 for $\tilde{\sigma}(b_0^1)$ into this equality and solving provides the optimal bid function

$$b_0^1 = \frac{v_0^1}{2} + \delta\bar{U}^I. \tag{26}$$

Thus, in the first round, agent's bid (and therefore the seller receives) all of the expected value of incumbency and one half of the current contact value. This result is in contrast to the previous case in which only one half of the value of the current contract is bid, and the winning bidder retains the full value of incumbency.

4 Conclusion

This article examines repeated auctions with the right of first refusal. A general form of the problem is first presented, and then optimal bid functions for first-period

[9]This solution corresponds to a constant of integration equal to zero and boundary condition $\tilde{\sigma}(c) - \delta\bar{U}^I = 0$, where c is an arbitrary constant. This condition corresponds to a constant of integration equal to zero. No other non-empty solutions were found for economically reasonable boundary conditions given the current problem.

bidders, entrants, and incumbents are derived for two different cases. The first case assumes that contract values differ across agents but not over time, and the second case assumes that contract value is stochastic across agents and over time for each agent. Although this difference in assumptions may at first glance appear innocuous, it requires substantially different solution approaches; especially to derive the optimal bids of entrants challenging an incumbent.

These two cases also highlight the value of information and its distributional effects. When agents know their private contract values into the future and the incumbents value of the contract, initial bidders choose to bid only one half of their initial period contract value, so if they win, they retain one half of the first period's contract value plus the entire expected value of incumbency beyond that. In contrast, when an agent's future valuations are stochastic such that all agents know each agent's distribution, each agent knows their value prior to the current auction, but no one knows their future contract values, then initial bidders bid away the entire value of future incumbency in addition to half of the first-period value. Thus, sellers are better off (and initial bidders are worse off) when agents do not know the incumbent's contract value.

When entrants know the incumbent's contract value, incumbents are unseated only by agents with higher contract values, but otherwise retain their initial bid of one-half of their contract value. In contrast, entrant's facing temporally stochastic contract values bid a fraction (between 1/2 and 3/4) of their net present value of winning in hopes that this bid will be higher than the net present value of the incumbent for that period.

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