Bid or Buy-it-now?
Time-Sensitivity and Price Dispersion in Online Retail Markets

Dominic Coey,* Bradley Larsen,† and Brennan C. Platt‡

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Abstract

We consider a population of buyers who have unit demand for a homogeneous good, and only differ in terms of how soon they need to purchase it. These buyers have access to a stochastic stream of second-price auctions, as well as posted-price listings that can be used at any time. Using the tools of equilibrium search theory, we characterize the equilibrium bidding dynamics, showing that bidders steadily raise their reservation price as they approach their deadline. This gives rise to an endogenous distribution of buyer valuations, and produces a considerable degree of dispersion in auction revenue. We also model the decision of sellers to list the item in an auction versus posted-price listings, and demonstrate that simple changes in auction design can create unexpected shifts in the distribution of buyer valuations.

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1 Introduction

Standard static auction models depict bidders as having a single opportunity to acquire the good in question, receiving a payoff of zero upon losing. While this seems appropriate for unique one-of-a-kind treasures, online auction sites frequently offer multiple listings of products that are readily available at retail outlets. On eBay, for instance, many items are sold new-in-box and identified by their SKU; with popular items, one can encounter dozens of identical offerings over the course of a week. Thus, losers of the current auction still have ample opportunity to obtain the same good, whether through subsequent auctions or outright purchase (at local stores, other internet vendors, or even on eBay itself through a posted-price "buy-it-now" listing).

*eBay Research labs, dominic.coey@gmail.com

†Department of Economics, Stanford University, bradjlarsen@gmail.com

‡Department of Economics, Brigham Young University, brennan_platt@byu.edu

With this setting in mind, we examine a set of over 8 million new products offered on eBay via auctions and posted-price listings. With various sources for each homogenous product, it is curious that such auctions still produce considerable dispersion in their final price (the interquartile range is typically 15% of the average closing price). Why do bidders differ in their willingness to pay when they share a common outside option? Moreover, a loser in a prior auction tends to bid more for the identical item in subsequent auctions. Why are bidders increasingly willing to pay more, even though identical auctions keep coming at a steady pace? Finally, the average auction closing price provides a modest 10% discount below the average posted-price sale. Why do sellers offer their inventory via auction when they could earn more using posted prices?

We rationalize these choices in a model of sequential auctions occurring in a continuous time search environment. Buyers have unit demand for a homogeneous item and encounter second-price sealed-bid auctions for the item at random intervals. They also have an option to purchase the item at any time at a posted price. In addition, each buyer faces a private deadline for their purchase. This can be interpreted literally — for instance, if the item is needed as a present for a child's birthday or to be used at a given time — but could also represent any other limit on how long a buyer is willing to attempt procurement via auctions.[1]

Deadlines introduce non-stationarity into the search problem, allowing bids to increase with the duration of search. At his deadline, the customer will purchase it at the posted price; but before the deadline, he might win the item in an auction for a lower price. Thus, his bid is shaded down from the true valuation, with more time providing more opportunity and hence greater shading. Even when anticipating a steady flow of auctions, a bidder will offer a higher price in each subsequent auction. This also explains why auction prices are necessarily lower than posted prices, which customers only use as a last resort.

Deadlines also generate rich price dynamics. To demonstrate this, we consider the stark case in which all buyers are identical in their valuation and their deadline.[2] Despite this ex-ante homogeneity, differences arise among them ex-post because some customers take longer than others to win an auction, leading to an endogenous, non-degenerate distribution of instantaneous bids, even though all buyers enjoy the same eventual utility from the good. This distribution, along with variation in the number of participants in each auction, allows

[1]A time limit could arise because the cost of auction participation increases, perhaps because customers cannot sustain the same level of attention to the auctions, or become increasingly frustrated with repeatedly losing auctions. Alternatively, the consequence of not winning could deteriorate with time; for instance, customers could be shopping for a replacement part (such as an engine timing belt) that hasn't yet failed but is increasingly likely to do so. Either way, customers anticipate a deadline beyond which it would be unwise to wait for additional auctions.

[2]Einav, *et al* (2013) and Hendricks, *et al* (2012) similarly presented models of eBay buyers with identical valuations for products.

the model to replicate the observed dispersion in auction closing prices.

We then close the model by having each seller endogenously determine the whether to list their unit in an auction or with a posted price. Despite their lower prices, auctions provide the advantage of selling immediately. If listed under a posted price, the item will be one of hundreds and may take weeks before that listing is randomly selected by customers hitting their deadline. If sellers were heterogeneous in their time preferences, the most patient among them would clearly opt for the posted prices; however, we demonstrate that a mixed strategy equilibrium can allow identical sellers to be indifferent between the two selling mechanisms. Thus, auctions and posted price listings can coexist, and even be offered by the same individual.

Most comparative statics are quite intuitive. For instance, more frequent auctions will benefit buyers by reducing search frictions; as a consequence, they shade their bids further and expected revenue falls. The surprising exception is when the deadline is extended for all bidders. One would expect this to also help a buyer by providing more opportunities before he must resort to the posted-price listings. However, this also increases the average number of buyer in the market; as a consequence, each auction is more likely to be competitive and expected revenue increases!

Market design takes on more nuance in our environment because changing the auction parameters will alter the distribution of valuations. We illustrate this with a simple increase in listing fees. Even with fees increased equally on both selling mechanisms, sellers shift from auctions to posted prices. As a consequence, auctions are less plentiful and thus less valuable to buyers, making them willing to pay more. Despite only being a concern for sellers, increased fees will compress the distribution of bids.

Our paper contributes to the competing mechanisms literature, which considers a seller's choice between auctions and posted price mechanisms. Julien, *et al* (2001), Einav, *et al* (2013), and Wang (1993) provide models in which one mechanism is strictly preferred over the other except in knife-edge or limiting cases. Etzion, *et al* (2006), Caldentey and Vulcano (2007), and Hammond (2013) present models in which both mechanisms coexist, relying on heterogeneity among buyer valuations or seller costs. The novel result of our model is that both mechanisms are employed in equilibrium, even though buyers and sellers are homogeneous ex ante.[3]

Our paper also contributes to the nacent literature on infinite sequential auctions (Zeithammer, 2006; Ingster, 2009; Said, 2011, 2012; Backus and Lewis, 2012; Bodoh-Creed,

[3]The above papers also differ from ours in that each assumes a static setting, where buyers only have one round in which to make their purchase. An exception to this is Kultti (1999) which examines a dynamic setting, obtaining a continuum of payoff-equivalent equilibria in which wait times are equivalent in auctions vs. posted prices. In contrast with the Kultti (1999) model, we find a unique equilibrium predicting that buyer wait times are shorter in posted-price listings than in auctions (and vice-versa for sellers), a finding consistent with eBay data, as discussed below.

2012; Hendricks, *et al*, 2012), in which bidders shade their bids below their valuations, taking into account the continuation value of participation in future auctions.[4] These papers, as well as ours, focus on dynamics between auctions rather than within an auction, which occur instantaneously via a second-price sealed bidding.[5] We contribute to this literature by providing an alternative source of heterogeneity among bidders. Rather than assuming buyers fundamentally differ in their valuations, which are drawn from an exogenous distribution, we assume all buyer are identical at the time they enter the market. This seems plausible in the context of auctioning standardized products.[6] Thus, differences arise endogenously because some unlucky bidders search longer than others, and therefore are nearer their deadline and willing to pay more. A distinguishing prediction of our model, and one for which we find strong empirical evidence, is that a bidder's bid will increase the longer the bidder has participated in auctions for a given item.[7]

Finally, our paper contributes to the literature in industrial organization seeking to explain equilibrium price dispersion, where search frictions allow sellers to charge distinct prices for a homogeneous good. This behavior is rational for both parties when buyers differ in their search costs (Salop and Stiglitz, 1976; Stahl, 1989; Sorensen, 2000; Schneider, 2014) or are inattentive (Malmendier and Lee, 2011); or sellers obfuscate (Ellison and Wolitzky, 2012) or do not honor previous quotes (Akin and Platt, 2014). In our model, buyers value the good identically, but their differing deadlines create a continuum of dispersed prices (as it did for labor markets in Akin and Platt, 2012). Also, the preceding papers are confined to retail sellers; we provide the first analysis of price dispersion in an auction environment.[8]

[4]This literature also focuses on online auctions in particular. Earlier work considered a finite sequence of auctions (Milgrom and Weber, 2000; Engelbrecht-Wiggans, 1994; Jeitschko, 1999), which induces a common deadline for all potential buyers, beyond which the good cannot be obtained.

[5]The exceptions are Hendricks, *et al* (2012), where new bidders move last to avoid disclosing their arrival, and Said (2012), where each period features a multi-unit ascending auction. Within-auction dynamics have mostly been studied in the context of a single auction (Nekipelov, 2007; Ambrus, *et al*, 2013) or concurrent auctions (Peters and Severinov, 2006; Ely and Hossain, 2009). In those environments, bidders can benefit from incremental bidding or waiting till the last minute (*sniping*) rather than submitting their true valuation as their only bid.

[6]A common valuation is also assumed in Hendricks, *et al* (2012). There, heterogeneity arises in an asymmetric strategy where newly-arrived bidders have an informational advantage and wait until all incumbent bidders (losers of prior rounds) have bid.

[7]In Said (2011) and Bodoh-Creed (2012), a bidder's valuation is redrawn between each auction; thus it is not surprising that the bid of those participating in their first auction would be distributed the same as the bid of those participating in their n^{th} auction. In Ingster (2009), a bidder holds the same valuation across all auctions and always submit the same bid; but since those who draw a high valuation are most likely to win, they will have shorter participation spells. That is, the average submitted bid would be lower among those who have been participating longer. Bidder valuations are also constant in Zeithammer (2006), Backus and Lewis (2012) and Hendricks, *et al* (2012), but bids adjust in response to other state variables (number of upcoming auctions, information about other past bidders, and number of past bidders, respectively). These states evolve in a Markov process, and would likely dampen any drift in bids over the participation spell.

[8]The model used in both Kultti (1999) and Julien, *et al* (2001) provides a first step toward price dispersion in an auction environment. A bidder wins for free if he is alone, but competes to the common valuation if anyone else participates. The random number of bidders also adds to dispersion in our model, but deadlines

In addition to these insights on generating equilibrium price dispersion, we employ two modeling features that are common in the search literature but largely unexploited in the auction literature: analysis in continuous time and steady state distributions. Specified in a continuous time framework, the model's equilibrium conditions can be translated into a solvable system of differential equations, which was the methodological innovation of Akin and Platt (2012). Here, the benefits extend beyond analytic tractability. Since buyers' bid functions depends on their state variable (time until the deadline), having a continuous state variable allows for a continuous distribution of buyer willingness to pay, as is typically assumed in static auction models. If our model were depicted in discrete time, there would be a mass of buyers at each state/bid level, and thus ties between bidders would be a real concern.[9]

We also restrict our analysis to steady state behavior; that is, the distribution of buyers (with respect to their deadline) remains stable. This keeps population dynamics tractable yet consistent with steady flows of incoming buyers and outgoing winners, and can best be interpreted as focusing on long-run behavior in a market fairly thick with buyers, which seems reasonable for the auctioning of standardized retail items. One of the benefits of employing these search-theoretic methods is greater tractability, allowing us to evaluate comparative statics readily, for instance. Most of the cited literature focus primarily on the structural estimation of the model and does not investigate how models reacts to parameter changes.[10]

We proceed by first discussing several motivating facts from eBay data in Section 2. We then develop the model for buyers and characterize its solution in Section 3, and then close the model in Section 4 by introducing the seller's problem and describing its equilibrium behavior. Section 5 presents a detailed analysis of our data and empirical predictions, and Section 6 concludes. All proofs are found in the Appendix.

2 eBay Background and Data

Although originally started as an experiment attempting to sell a single used, broken lazer pointer, eBay has become a large marketplace for millions of new-in-box items. The platform offers sellers the choice of several sales mechanisms, with the most popular being a fixed

are the dominant force, as we show in Section 3.5.

[9]Relative to the sequential auctions literature, only Ingster (2009) uses a continuous time model, but as bidder heterogeneity is exogenous there, our approach is unnecessary for that solution. Rather, the solution to the discrete time model in Said (2011) has more parallels to our method; there, the relevant state variable affecting bid shading is the (discrete) number of active bidders. The equilibrium conditions are then depicted as a first order system of difference equations.

[10]The exceptions are Zeithammer (2006) and Said (2011), which feature comparative statics with respect to the discount rate, the time between auctions, and (in the latter) the expected number of competitors. Reservation prices respond to these in the same direction as in our model.

price (“Buy-It-Now“) or a second-price-like auction.[11] Despite the homogeneity of many of these goods, there exists substantial variation in prices. As an example, Figure 1 displays the results from a search for sold auction listings for brand new LEGO Millenium Falcon sets, a popular childrens toy. Within a period of less than 48 hours, six auctions of this item ended, with final prices ranging from \$98.76 to \$121.37.[12] In contrast, this item was offered for sale at a fixed price of \$138.99 at the same time on Amazon.com. These facts raise the questions of why price dispersion exists and why a seller would choose to sell an item through an auction rather than at a higher fixed price.

3 Model

3.1 Buyers

Consider the market for a homogeneous good in a continuous-time environment. Buyers randomly enter the market at Poisson rate δ, needing to obtain one unit of the good in T units of time. Each buyer enjoys the same utility x (measured in dollars) from the good, which is split[13] between the immediate value from purchase, βx, and the value realized only at the deadline, $(1-\beta)x$. The latter is discounted at rate ρ. Thus, if the good is purchased with s time remaining until the buyer's deadline, his realized utility is $\left(\beta + (1-\beta)e^{-\rho s}\right)x$ minus the purchase price.

The good is offered in second-price sealed-bid auctions that occur at Poisson rate α. When such an auction occurs, each buyer in the market participates with exogenous independent probability τ, reflecting that buyers can be distracted by other commitments. Each participant submits a bid and immediately learns the auction outcome, with the highest bidder winning and paying the second highest bid. Alternatively, at any time, a buyer can obtain the good directly at a posted price of z, provided by traditional retailers, by third-party sellers on Amazon, or as posted-price listings on eBay.

[11] For a discussion the eBay auction mechanism, see (??). Not included in our analysis is a hybrid of these two formats, called the *buy-it-now auction*, in which the seller lists the item in an auction but also offers a buy-out price. The auction proceeds as normal unless a bidder exercises the buy-out price, who then pays the price and wins the item, while all other auction participants leave empty handed (e.g. Budish and Takeyama, 2001; Kirkegaard and Overgaard, 2008; Bauner, 2013).

[12] These prices could also differ for a number of reasons not shown in the figure, such as seller rating or shipping costs. We demonstrate below that price dispersion exists even with a given seller and conditional on the shipping fee.

[13] The extreme of $\beta = 0$ indicates that the good is literally of no use until the date of the deadline, while $\beta = 1$ indicates that it starts providing the same flow of value regardless of when it is purchased. The intermediate case seems reasonable for many deadlines: for instance, a gift is not needed until the birthday, but the giver may enjoy some peace from mind of having it secured early. A spare automobile part provides similar insurance even if it is not literally needed until the failure of the part it replaces. A home in a new city may not be needed before employment starts there, but early acquisition allows for a smoother transition.

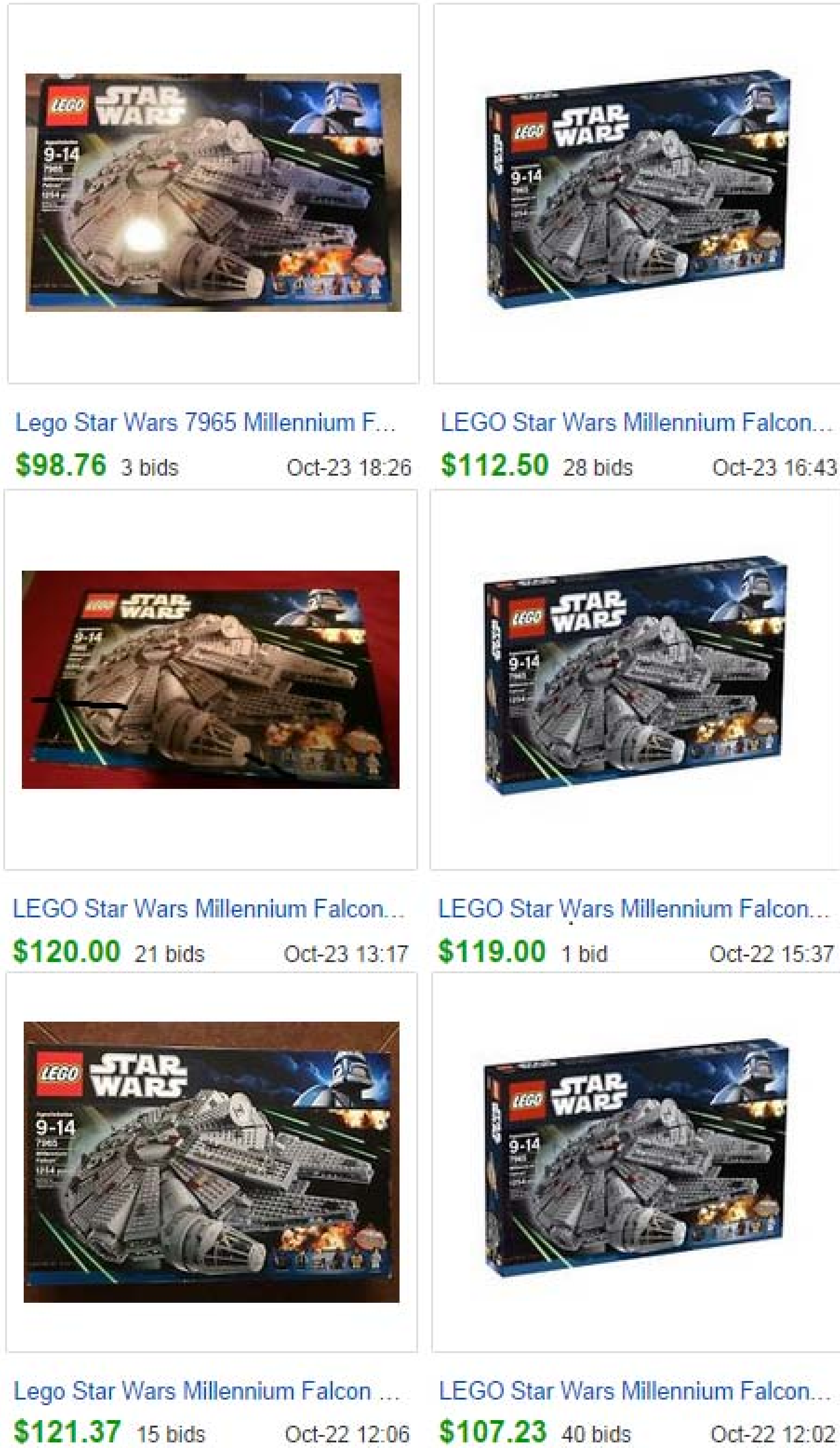

Figure 1: Screen shot of eBay search for phrase "millenium falcon lego" for recently sold auction listings of new items (retrieved 10-23-2014).

We assume throughout that $x \geq z$, so that buyers weakly benefit from purchasing via the posted-price option. We further assume that $\beta x < z$; otherwise, buyer would strictly prefer using the posted-price option in equilibrium, as the possible savings from winning at auction cannot compensate for delaying the immediate benefits of an outright purchase.

Every buyer shares the same utility x and deadline T on entering the market; but because they randomly enter the market, they will differ ex-post in their remaining time s. In any given auction, the number of bidders and their state s are private information. However, the distribution of bidders in the market (represented by cumulative distribution $F(s)$) is commonly known, as is the average number of buyers in the market, which is Poisson distributed with mean H. Both $F(s)$ and H are endogenously determined. Note that the Poisson processes governing the entry of buyers (δ), the stock of buyers (H), and the arrival of auctions (α) are all independent but will be related to one another in equilibrium by the steady state conditions in the next subsection.

The strategic question for buyers is what bid to submit and when to purchase from the posted-price listings. This dynamic problem can be expressed recursively, letting $V(s)$ denote the discounted expected utility of a buyer with s time remaining until deadline. Each participant submits a bid $b(s)$ that depends on his remaining time until deadline. In particular, each auction is independent, since it draws a new set of participants from the pool of buyers.[14]

The optimal behavior is to bid one's reservation value, setting:

$$b(s) = \left(\beta + (1-\beta)e^{-\rho s}\right) x - V(s); \tag{1}$$

that is, the present value of the item minus the opportunity cost of skipping all future auctions. As in the standard second-price sealed-bid auction, this strategy is weakly dominant regardless of that employed by other participants.[15] We assume that $b(s)$ is decreasing in s (that is, willingness to pay increases as the deadline approaches), and later verify that this holds in equilibrium. The auctioneer is assumed to open the bidding at $b(T)$, which is relevant in the case that only one buyer participates; we consider seller optimization of the reserve price in Section 4.7.

[14] One abstraction in our model is that we assume that bidders do not infer any information about their rivals from prior rounds. Such information is unimportant if valuations are redrawn between each auction, but if valuations are persistent, this *leakage* of private information can hinder future success and leads to further shading of bids (Backus and Lewis, 2012; Said, 2012). In a large market, though, the cost of recording hundreds of bidders' past actions seems impractical; also, since not every bidder participates in every auction, the expected value of such information diminishes quickly. Zeithammer (2006) uses the same assumption and justification.

[15] Suppose he instead bids price $p > b(s)$, and the second highest bid is q. This results in the same payoff whenever $q \leq b(s)$, but yields negative surplus when $q \in (b(s), p]$. Similarly, if he bids $p < b(s)$, he has the same payoff whenever $q \leq p$, but misses out on positive surplus when $q \in (p, b(s))$.

Since fraction τ of buyers participate in a given auction, the number of participants per auction is Poisson distributed with mean $\lambda = \tau H$; that is, the probability that n bidders participate is $e^{-\lambda}\lambda^n/n!$. While this literally governs the *total* number of participants per auction, the same density function also describes (from the perspective of a bidder who has just entered) the probability that n *other* bidders will participate. This convenient parallel between the aggregate distribution (which enters expected revenue and the steady state conditions) and the distribution faced by the individual (which enters his expected utility) is crucial to the tractability of the model but is not merely abuse of notation. Myerson (1998) demonstrates that in Poisson games, the individual player would assess the distribution of other players the same as the external game theorist would assess the distribution for the whole game.

In light of this, a buyer's expected utility in state s can be expressed in the following Bellman equation:[16]

$$\begin{aligned} \rho V(s) = -V'(s) + \tau\alpha \Bigg(& \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}(1-F(s))^n \left(\left(\beta + (1-\beta)e^{-\rho s}\right)x - V(s)\right) \\ & - e^{-\lambda}b(T) - \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \int_s^T b(t)n(1-F(t))^{n-1}F'(t)dt \Bigg). \end{aligned} \tag{2}$$

In this continuous-time Bellman equation, the left-hand side represents the flow of expected utility that a buyer with s time remaining receives each instant. On the right hand side, we depict any potential changes in (net) utility times the rate at which those changes occur.[17] For example, the term $-V'(s)$ accounts for the steady passage of time: just by remaining in the market for another unit of time, the buyer's state s decreases by 1 unit (hence the negative sign) and his utility changes by $-V'(s)$.

When an auction occurs and the individual participates in it — which occurs at a rate of $\tau\alpha$ auctions per unit of time — the expected payoff depends on the number n of other participants, which is Poisson distributed with mean λ. The buyer in state s will only win (have the highest bid $b(s)$) if all n other participants have more than s time remaining; this occurs with probability $(1-F(s))^n$. When this occurs, the term $\left(\beta + (1-\beta)e^{-\rho s}\right)x - V(s)$ depicts the change in utility due to winning.

The terms on the second line compute the expected cost of winning (*i.e.* the average second-highest bid times the probability of winning and thus paying it). If there are no other participants (which occurs with probability $e^{-\lambda}$), the bidder pays the starting price $b(T)$.

[16] A discrete time formulation of this Bellman equation is provided in the appendix.

[17] If buyers received any enjoyment from the search process itself, it would appear as a positive constant on the left-hand side. This would have negligible effect on the solution other than lowering willingness to pay, since making a purchase would cut off this flow of enjoyment.

Otherwise, inside the sum we find the probability of facing n opponents, while the integral computes the expected highest bid among those n opponents, which has a probability density of $n(1-F(t))^{n-1}F'(t)$.[18]

Buyers also have the option to purchase from the posted-price listings at any time, receiving utility $(\beta + (1-\beta)e^{-\rho s})\,x - z$. However, a buyer in state s can obtain a discounted expected utility of $e^{-\rho s}(x-z)$ by waiting until $s=0$ to make the purchase, and this is strictly preferred assuming that $\beta x < z$. This delay strategy has even greater payoff due to the possibility of winning an auction in the meantime. Hence, the posted-price option is exercised if and only if $s=0$, and the expected utility of a buyer who reaches his deadline is simply the consumer surplus from making the purchase:

$$V(0) = x - z. \tag{3}$$

3.2 Steady State Conditions

In most auction models, this distribution of willingness to pay is exogenously given as a primitive of the model. Here, the buyers' reservation prices $b(s)$ are endogenous, affected by the value of further search ($V(s)$ in Equation 1) in addition to the underlying utility x. More importantly, the distribution $F(s)$ of buyer states is also endogenously determined by how likely a bidder is to win and thus exit the market at each state, which itself depends on the distribution of competitors he faces.

We require that the distribution of buyers remains constant over time. As buyers exit the market, they are exactly replaced by new customers; as one group of buyers get closer to their deadline, a proportional group follows behind. These steady state requirements are commonly used in equilibrium search theory to make models tractable. Rather than tracking the exact composition of current market participants, which would stochastically shift with each entry or exit and thus require a large state space, the aggregate state is always held at its average. This does not eliminate all uncertainty — for instance, the number of bidders in a given auction need not equal the average λ — but these shocks are transitory, as the number of bidders in the next auction is independently drawn from a constant (though endogenous) Poisson distribution. Thus, steady state conditions smooth out the short-run fluctuations around the average, and are interpreted as capturing the long-run average behavior in a market.[19]

[18] Note that since the range of integration is only from s to T, $n(1-F(t))^{n-1}F'(t)$ would only sum to $(1-F(s))^n$ rather than 1. This coincides with the probability that this buyer's bid is higher than all opponents, as required.

[19] Of course, this prevents bidders from conditioning their strategy on who they faced in prior auctions, which Backus and Lewis (2012) consider with valuations drawn from an exogenous distribution. This feature would make it impossible to solve for our endogenous distribution, but in a large market with hundreds of potential bidders, bidders would have trouble tracking the aggregate state and would not adjust their strategy

To begin, consider the relative density of bidders over their time until deadline. For instance, consider a cohort in state $s > 0$. In the next Δ units of time, suppose on average that a fraction w of these buyers win an auction and exit each unit of time. Then steady state requires that $F'(s-\Delta) = F'(s) - w\Delta F'(s)$. After rearranging and letting $\Delta \to 0$, we obtain $F''(s) = wF'(s)$. Formally, the steady-state law of motion is expressed as follows:

$$F''(s) = F'(s)\tau\alpha \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}(1-F(s))^n. \tag{4}$$

Here, $\tau\alpha$ is the rate at which buyers participate in an auction, and the summation indicates how likely they are to have the highest bid of n participants and thus win. Remember that all bidders enter at $s = T$; at all other states, some bidders exit. Thus the bidder density $F'(s)$ must decrease as s falls, which holds formally because the right-hand side of Equation 4 is always positive.

Equation 4 defines the law of motion for the interior of the state space $s \in (0, T)$. The end points are defined by requiring $F(s)$ to be a continuous distribution:

$$\lim_{s\to 0} F(s) = F(0) \quad = \quad 0 \tag{5}$$
$$\lim_{s\to T} F(s) = F(T) \quad = \quad 1. \tag{6}$$

While any cumulative distribution function must remain between 0 and 1, these two conditions prevent a discontinuous jump at either end of the distribution — that is, a positive mass (an *atom*) of buyers who share the same state, $s = 0$ or T. Atoms cannot occur in our environment because all buyers who reach state $s = 0$ immediately purchase from a posted-price listing and exit the market; hence, no stock of state 0 buyers can accumulate. Similarly, no stock of state T buyers can accumulate because as soon as they enter the market, their clock begins steadily counting down. Conveniently, a continuous distribution ensures that no two bids will tie with positive probability.

Finally, we ensure that the total population of buyers remains steady. Since H is the average number of buyers in the market, $HF(s)$ depicts the average number of buyers with less than s time remaining, and $HF'(s)$ denotes the average flow of state s buyers over a unit of time. Thus, we can express the steady state requirement as:

$$\delta = H \cdot F'(T). \tag{7}$$

Recall that buyers exogenously enter the market, averaging δ new buyers in one unit of time. Since all buyers enter the market in state T, this must equal $HF'(T)$, the average

much even if they did.

number of state T buyers in one unit of time.

3.3 Buyer Equilibrium Definition

The preceding optimization by buyers constitutes a dynamic game. We define a *buyer steady-state equilibrium* of this game as a bid function $b^* : [0,T] \to \mathbb{R}$, a distribution of buyers $F^* : [0,T] \to [0,1]$, an average number of buyers $H^* \in \mathbb{R}^+$, and an average number of participants per auction $\lambda^* \in \mathbb{R}^+$, such that:

1. Bids b^* satisfy the Bellman Equations 1 through 3, taking F^* and λ^* as given.
2. The distribution F^* satisfies the Steady State Equations 4 through 6.
3. The average number of active buyers H^* satisfies Steady State Equation 7.
4. The average number of participants per auction satisfies $\lambda^* = \tau H^*$.

The first requirement requires buyers to bid optimally; the last three require buyers's beliefs regarding the population of competitors to be consistent with steady state.

3.4 Buyer Equilibrium Characterization

We now present the unique equilibrium of this sequential auctions of a homogeneous retail good. Our equilibrium requirements can be translated into two second-order differential equations regarding $F(s)$ and $b(s)$. The equations themselves have an analytic solution, but one boundary condition does not have a closed-form solution. Rather, we solve for the equilibrium λ^* which implicitly solves the boundary condition. If we define $\phi(\lambda)$ as:

$$\phi(\lambda) \equiv \alpha\left(1 - e^{-\lambda}\right) - \delta + \delta e^{\lambda - \tau T\left(\delta + \alpha e^{-\lambda}\right)}, \tag{8}$$

then the boundary condition is equivalent to $\phi(\lambda^*) = 0$, which ensures that no atom of buyers occurs at state $s = 0$. Thus, the rest of the equilibrium solution is expressed in terms of λ^*.

First, the distribution of buyers over remaining time until deadline is:

$$F^*(s) = 1 - \frac{1}{\lambda^*} \ln\left(\frac{\delta + \alpha e^{-\lambda^*}}{\delta e^{\tau(s-T)\left(\delta + \alpha e^{-\lambda^*}\right)} + \alpha e^{-\lambda^*}}\right). \tag{9}$$

Intuitively, the density function F' is strictly increasing in s, since some buyers win auctions and thus exit as the clock s counts down. The rate of increase depends on s, typically changing from convex to concave as s increases. This is because buyers rarely win at the beginning of their search, but increasingly do so as time passes and they increase their bids.

Those near their deadline win quite frequently, but few of them remain in the population, so their rate of exit decelerates.

The average number of buyers in the market is simply:

$$H^* = \frac{\lambda^*}{\tau}. \tag{10}$$

Equilibrium bids are expressed as a function of the buyer's state, as follows:

$$b^*(s) = z - (z - \beta x) \frac{\delta \left(\alpha\tau + (\delta\tau + \rho)e^{\lambda^*}\right)(1 - e^{-\rho s}) + \alpha\rho e^{\tau T\left(\delta + \alpha e^{-\lambda^*}\right)}\left(1 - e^{-s\left(\rho + \tau\left(\alpha e^{-\lambda^*} + \delta\right)\right)}\right)}{\delta\rho e^{\lambda^*} + \tau\left(\delta + \alpha e^{-\lambda^*}\right)\left(\delta e^{\lambda^*} + \alpha e^{-\rho T}\right) + \alpha\rho e^{\tau T\left(\delta + \alpha e^{-\lambda^*}\right)}}. \tag{11}$$

The following result establishes that this proposed solution is both necessary and sufficient to satisfy the equilibrium requirements. The proof (provided in the appendix) is constructive, translating the necessary equilibrium conditions into differential equations of $F(s)$ and $b(s)$. These equations each have unique solutions, leading to the sufficiency result.

Proposition 1. *Equations 9 through 11 satisfy equilibrium conditions 1 through 4, and this equilibrium solution is unique.*

As previously conjectured, one can readily show that $b'(s) < 0$; that is, bids increase as buyers approach their deadline. Moreover, this increase accelerates as the deadline approaches, since $b''(s) > 0$.

Proposition 2. *In equilibrium, $b'(s) < 0$ and $b''(s) > 0$.*

Discounting plays a critical role in creating dispersion among the bidder valuations. For instance, if buyers become extremely patient ($\rho \to 0$), the bidding function approaches $b(s) = z$ regardless of time until deadline.[20] In other words, prices become less dispersed as patience increases, all else equal.

The average time between auctions ($1/\alpha$) is of similar importance. In effect, this is the search friction that buyers face, as it prevents them from making unlimited attempts at winning an auction. In the extreme, if auctions almost never occurred ($\alpha \to 0$), the value of search $V(s)$ drops to zero,[21] so a bidder's reservation price would simply equal his present value of the good: $b(s) = (\beta + (1 - \beta)e^{-\rho s})\, x$. In contrast, for larger values of α, a bidder would optimally reduce his bid well below this, since he is likely to have several opportunities to win a deal before his deadline.

[20] The fractional term of Equation 11 approaches zero. Note that the equilibrium λ^* is unaffected (Eq. 8), as is the distribution of bidders (Eq. 9).

[21] In this limit, the equilibrium $\lambda^* = \tau\delta T$, while the is the distribution of bidders is $F(s) = s/T$.

3.5 Calibration

The model can be easily calibrated to match the facts provided in Section 2 regarding eBay auctions. We illustrate the equilibrium behavior under these parameters, but they are also representative of what occurs over a large range of parameterizations.

We begin by normalizing the posted price $z = 1$. This has no effect on the distribution $F(s)$, and as long as x is scaled proportionally, then $R(s)$ will also scale proportionally. We similarly transform bidding data: for each item, we computing the average price among all sold posted-price listings (the analog of z), then divide all bids for that item by this average. We also consider one unit of time to be a month.

Other parameters are then computed as depicted in Table 1. We use completed auctions (i.e. in which at least one bidder arrived) for our calibration targets rather than listed auctions because transactions in the latter require rational choices by both buyer and seller, while auctions that are listed but not sold could have failed due to bad luck (as in the model) or bad seller decisions (like setting the opening price too high).

The first four items in Table 1 are either directly observed or require minimal adjustment to extract from the data.[22] Note that participants per auction, λ, is endogenous but observable. By assuming this participation is consistent with equilibrium, we can back out the deadline length T from the equilibrium condition that $\psi(\lambda) = 0$ (Eq. 8).

The final two items in Table Table 1 use two moments of the price distribution (whose formulas are reported in the Appendix with the proof of Proposition 3) to pin down the discount rate and immediate utility from purchase, and must be jointly solved with numeric methods. Also, note that we cannot separately identify β from x, as these two parameters are always appear multiplied together in any of our equilibrium conditions.

On their face, the resulting parameters are plausible, though the discount rate is a bit higher than the typical interest rate might dictate. Note that, by calibration, the average auction generates revenue equal to 98.2% of the average posted price. Fifteen percent of buyers are unsuccessful in purchasing via the auctions and eventually use the posted-price option.

For a more rigorous validation of the calibration, we can examine two dimensions not used in the calibration. First, although we have used two moments of the distribution of prices to calibrate ρ and βx, we have not used any other details regarding the shape of the distribution. That is, we know that the distribution is slightly skewed towards higher bids (because the mean exceeds the median), but we have not used any details about the

[22] For the fourth row of Table 1, we compute the average number of bidding attempts per bidder across all auctions, conditional on having made at least n attempts. The model predicts this will follow the Poisson distribution $(\sum_{k=n}^{\infty} ke^{-\tau\alpha T}(\tau\alpha T)^k/k!)/(\sum_{k=n}^{\infty} e^{-\tau\alpha T}(\tau\alpha T)^k/k!)$, which we use to solve for $\tau\alpha T$. While this solution should be the same for all n, in the data it rises from 0.73 for $n = 1$ to 6.02 for $n = 6$, after which it stabilizes. We take the average of these estimates across $n = 1$ to 8.

Table 1: Calibration process and parameter values

Fact	Observed Value	Theoretical Equivalent	Calibrated Parameter
Bidders per completed auction	5.5	$\lambda / \left(1 - e^{-\lambda}\right)$	$\lambda = 5.48$
Completed auctions per month	30.2	$\alpha \left(1 - e^{-\lambda}\right)$	$\alpha = 30.3$
Auctions a bidder tries per month	1.29	$\tau\alpha$	$\tau = 0.043$
Auctions a bidder tries ever	4.69	$\tau\alpha T$	$T = 3.60$
—	—	Eq. 8	$\delta = 43.4$
Average revenue per completed auction	0.843	θ	$\rho = 0.067$
Median revenue per completed auction (not matched, currently 0.844)	0.92	ω	$\beta x = 0$

shape of the density. Thus, we compare the empirical distribution of bids to that predicted by the theory under these parameters. This is illustrated in Figure 2 and tested using both the Pearson's χ^2 test for goodness-of-fit and the KolmogorovSmirnov (K-S). Even if the sample and theoretical distributions have similar means, these distributional tests will reject equality if the relative densities (i.e., shape of the distributions) differ by too much in any particular region. These report a p-value of XX and XX, respectively, suggesting that predictions are somewhat close to the observed distribution.

Second, the calibration process did not exploit any details about the bids of an individual over time; even the moments of the distribution were across all auctions and all bidders. Under the calibrated parameters, the theory predicts that an individual will increase his bid at a rate of 6.7% per month, on average. Since the average bidder participates in 1.29 auctions per month, that translates to increase of 5.2% between each auction of a given item. In the data, we see an increase of XX between each auction attempt, with a confidence interval of XX.

The left panel of Figure 2 illustrates the equilibrium density of bidders. Note that $F'(s)$ is nearly constant from $s = 2.5$ to 6. Due to the large number of bidders per auction, those with lower valuations (hence longer time remaining) are highly unlikely to win. Ye the relative density cuts in half between $s = 2.5$ and $s = 0.8$, and then does so again before $s = 0$. Those closest to their deadline are far more likely to win and exit.

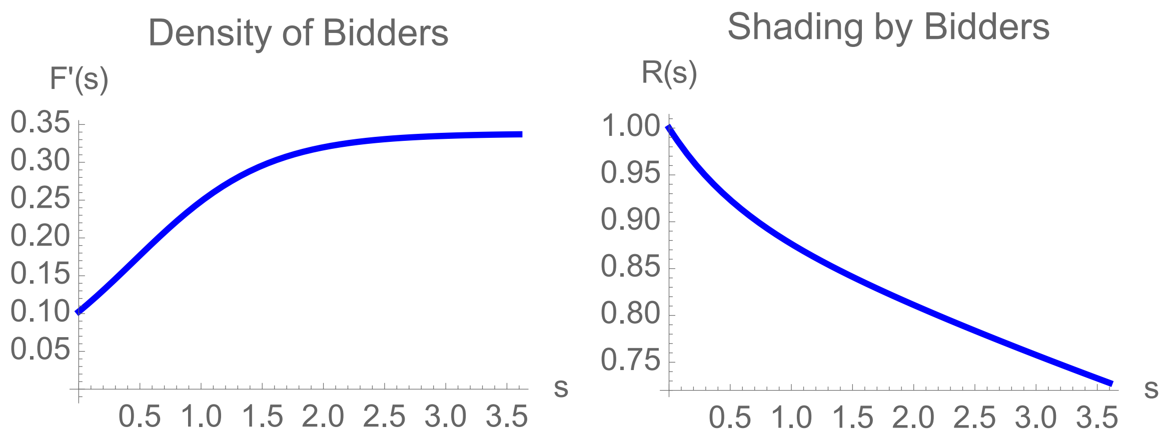

Figure 2: Numerical Example: the distribution of bidder states (left panel) and the % by which bidders shade relative to z (right panel)

The right panel of Figure 2 depicts the equilibrium path of bids. Since $z = 1$, these can be read as the factor by which bidders shade their bids below the posted price. Note that prices are dispersed across a range equal to 4% of the posted price. This becomes larger with higher discounting or longer deadlines. Initially (near $s = T$), the price path is more or less linear; but as the deadline approaches, greater curvature is introduced.

Figure 3 provides another perspective on the realized bids in the auction. The dot-dashed line plots the density of submitted bids in the typical auction. That is, for any price p on the x-axis, the y value indicates the relative likelihood of that price being placed as a bid. Effectively, this is $F'\left(b^{-1}(p)\right) \cdot (b^{-1})'(p)$, obtained via a parametric plot since $b^{-1}(p)$ cannot be analytically derived. This plot shares much in common with the plot of $F'(s)$ in the left panel of Figure 2, reversed in direction since the highest bids come from the bidders closest to their deadline.

This is contrasted with the dotted line, which plots the density of the highest bid in each auction. Note that this is heavily concentrated on the highest prices, even though bidders with those valuations are somewhat scarce. This is because of the large number of participants per auction — the highest of 13 bids tends to be close to the top of available bids.

Of course, only the second-highest price is actually paid; this density is depicted with the solid line. While this is less skewed towards high prices, it still provides reasonably high prices, with a modal closing price of 98.7% of z, which is approximately the reservation price of a bidder with $s = 1$. Note also the considerable dispersion of these closing prices, with an interquartile range of 1.5%.

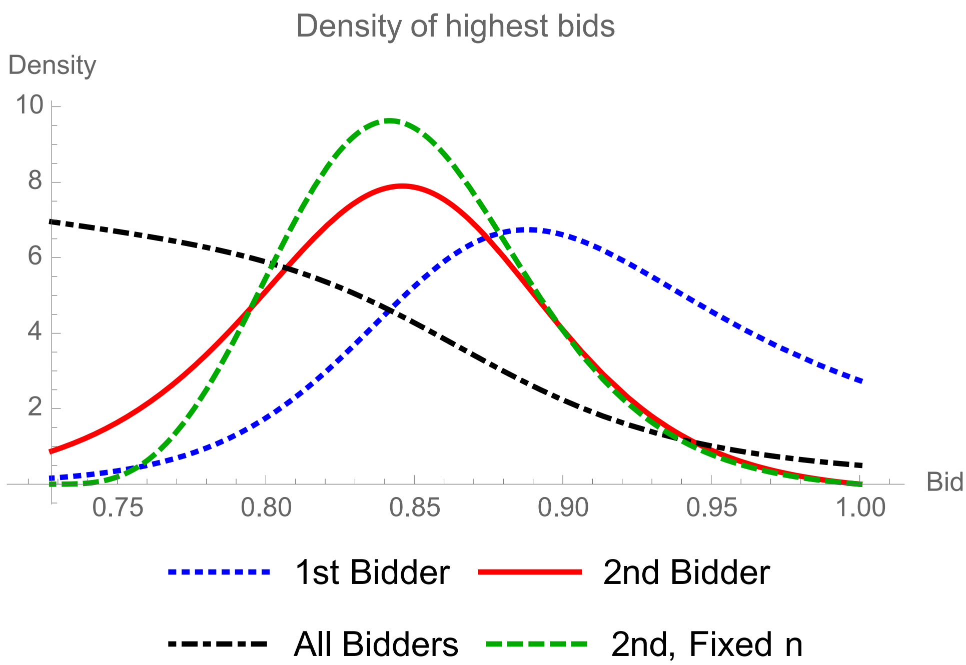

Figure 3: Price Density: the equilibrium density of the highest bid in an auction (dotted), the second highest bid (solid), and all bids (dot-dashed). Also pictured is the density of the second highest bid if the number of bidders were constant at $n = \lambda^*$.

Table 2: Comparative statics on key statistics: Buyer Equilibrium

		$\partial/\partial\alpha$	$\partial/\partial\tau$	$\partial/\partial\rho$	$\partial/\partial x$	$\partial/\partial\beta$	$\partial/\partial T$
Participants per Auction	λ^*	–	+	0	0	0	+
Buyers using Posted Price	$F'(0)H^*$	–	–	0	0	0	–
Lowest Bid	$b^*(T)$	–	+	–	+	+	–

Note: Derivations are provided in the appendix.

Auction revenue faces two sources of randomness: the number of participants and the distribution of how desperate they are. To separate these two effects, we consider what would happen if we eliminated the first. That is, what if the auction had exactly λ^* participants, rather than randomly drawing them (with λ^* participants on average)? We take these λ^* draws from the equilibrium distribution $F'\left(b^{-1}(p)\right)$, and plot the density of the second highest of these draws in the dashed line. By comparing this with the solid line, we conclude that the randomness in number of participants has negligible effect in the overall dispersion of closing prices, only reducing the standard deviation by 0.05%.

3.6 Comparative Statics

We next examine how the equilibrium behavior reacts to changes in the underlying parameters. Although our equilibrium has no closed form solution, these comparative statics can be obtained by implicit differentiation of the $\phi(k)$, which allows for analytic derivation which are reported in the technical appendix to the paper.

Table 2 reports the sign of the derivatives of three key statistics. First is the average number of participants per auction, λ^*, which reflects how competitive the auction is among buyers. Note that the average number of buyers in the market, H^* is always proportional to λ^*. Second is the flow of buyers who never win an auction and thus resort to the posted-price listings; in the next section we will see that this crucially affects the profitability of the posted-price market. Third is the bid of new buyers in the market, indicating the effect on buyers' willingness to pay. This comparative static can be derived at any s and has a consistent effect, but the computation is easiest to report at $s = T$.

Changes in α have a very intuitive impact. If auctions arrive more frequently, this reduces search frictions; that is, the value of continued search is greater as there are more opportunities to bid. Moreover, the increase in auctions will reduce the stock of bidders and hence the number of competitors per auction. Both of these effects lead bidders to lower reservation prices.

Changes in τ have nearly the reverse effect, though here there are opposing forces at work. Having a higher likelihood of participating also reduces the search friction of a given bidder, as he will participate in more of the existing auctions. On the other hand, all other bidders are more likely to participate as well. This greater number of competitors dominates the increased auction participation to reduce the value of search, increasing bidders' reservation prices.

The discount rate, the value of the good, and the fraction of immediate consumption utility all have no impact on the number or distribution of bidders, which can be seen mathematically in the fact that these parameters do not enter into Equations 8 through 10. Intuitively, this is because the rate at which bidders exit is a matter of how often auctions occur, which is exogenous here. Also, who exits is a matter of the ordinal ranking of their valuations, which does not change even if the cardinal values are altered. Indeed, the bids react as one would expect: buyers offer more when their utility from consumption is higher or more immediate.

We can also consider the effect of the parameter change on the expected revenue generated in an auction, which we formally derive in the next section. In all the preceding comparative statics, revenue moves in the same direction as bids because the number of participants per auction is either constant or moves in the same direction. For instance, more auctions will reduce the bids and reduce the number of bidders; thus expected revenue must be lower.

The intriguing exception is when the deadline changes. Intuitively, one would expect that an increase in T would work to buyers' advantage. Indeed, bids are lower, yet at a given s the change is minor; the larger impact is because T is further from the deadline so $b(T)$ is lower. At the same time, the longer period of search allows for more bidders to accumulate (H^* increases). Thus, auctions are more likely to be competitive. When participation per auction is somewhat low initially, this competition effect can dominate the falling bids, driving expected revenue up.

4 Seller Incentives

We next examine optimization by sellers in this environment, allowing them to decide whether to enter the market and whether to sell their product via auctions or the posted-price listing. We consider a continuum of sellers producing an identical good. Each has negligible effect on the market, taking the behavior of other sellers and the distribution and bidding strategy of buyers as given; yet collectively, their decisions determine the frequency with which auctions occur. In other words, by modeling seller choices we endogenously determine α from the preceding section.

Each seller can produce one unit of the good at a marginal cost of $c < z$, with fraction γ

of this cost incurred at the time the good is sold (the *completion cost*), and $1-\gamma$ incurred when seller first enters the market (the *initial production cost*).[23] For either selling format, sellers also pay a listing fee of ℓ each unit of time from when the seller enters the market to when the good is sold. We assume that there are no barriers to entry for sellers. Upon entry, each seller must decide whether to join the auction or the posted-price market.

4.1 Auction Sellers

The advantage of the auction sector is that the sale occurs more quickly. Let η denote the exogenous rate of auction closing, so $1/\eta$ is the average time delay between the listing and closing of an auction.[24] At its conclusion, the auction's expected revenue (conditional on at least one bidder participating) is denoted θ and computed as follows:

$$\theta \equiv \frac{1}{1-e^{-\lambda}}\left(\lambda e^{-\lambda} b(T) + \sum_{n=2}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \int_0^T b(s) n(n-1) F(s)(1-F(s))^{n-2} F'(s) ds\right). \quad (12)$$

Inside the parentheses, the first term applies when only one bidder participates and therefore wins at the opening price of $b(T)$; this happens with probability $\lambda e^{-\lambda}$. The sum handles cases when there are $n \geq 2$ simultaneous bidders, with the integral computing the expected bid $b(s)$ of the second-highest bidder. All of this must be divided by the probability that at least one bidder arrives, $1-e^{-\lambda}$.

To determine the expected profit of the auction seller, we must also account for the costs of production and the expected time delay. Let Π_a denote the expected profit from the vantage of someone who has already incurred the initial production cost $(1-\gamma)c$ and has just posted the listing. This profit can be computed in the following Bellman equation:

$$\rho \Pi_a = -\ell + \eta\left(1-e^{-\lambda}\right)(\theta - \gamma c - \Pi_a). \quad (13)$$

On the right-hand side, this indicates that the seller incurs the listing fee per unit of time. The listing closes at Poisson rate η, but if no bidders have arrived (which occurs with probability $e^{-\lambda}$) then the seller re-lists the item and continues waiting for the new auction to close. If at least one bidder participates, the seller's net gain is the realized benefit (revenue minus the completion cost) relative to the expected profit. Of course, from the perspective of a potential entrant as an auction seller, the expected profits from entry are

[23] In the extreme, $\gamma = 1$ would indicate the ability to build-to-order or just-in-time inventories, while $\gamma = 0$ indicates a need to build in advance (like a spec home built without a committed buyer). Intermediate values could be take literally as partial production, or as full initial production followed by additional expenses (such as shipping costs) at the time of sale. It could also reflect producing in advance but delaying full payment of the cost through the use of credit.

[24] Under the default setting at eBay, an auction concludes one week after creating the listing, which provides time for bidders to examine the listing. In our sample, XX% of listings use this default setting.

net of the initial production cost: $\Pi_a - (1-\gamma)c$.

4.2 Posted-Price Sellers

The posted-price listing will always sell the good at a higher price, since for all bids, $b(s) < z$; the disadvantage of this format is that sellers may wait a considerable time before being chosen by a seller. Let ζ denote the rate of encountering a customer, so $1/\zeta$ is the average wait of a posted-price seller. Sellers take ζ as given, but it will be endogenously determined as described in the next subsection.

The discounted expected profit of posted-price sellers already in the market, denoted Π_p, is computed in the following Bellman equation:

$$\rho\Pi_p = -\ell + \zeta\left(z - \gamma c - \Pi_p\right). \tag{14}$$

Like auction sellers, posted-price sellers incur the listing fee each instant.[25] When they encounter a buyer (which they do at rate ζ), the purchase always occurs, with a net gain of $z - \gamma c$ relative to Π_p. For sellers contemplating entry into the posted-price market, their expected profit is $\Pi_p - (1-\gamma)c$.

4.3 Steady State Conditions

As with the population of buyers, the stock and flow of sellers are also assumed to remain stable over time. In the aggregate, recall that δ buyers enter (and exit) the market over a unit of time; thus, we need an identical flow of δ sellers entering per unit of time so as to replenish the δ units sold.[26]

In addition, the number of sellers in each market must remain steady. For instance, if fraction σ of newly-entered sellers join the auction market, then $\sigma\delta$ choose to list an auction over a unit of time. This must equal the number of auctions that close with at least one bidder over the same unit of time:

$$\sigma\delta = \alpha\left(1 - e^{-\lambda}\right). \tag{15}$$

The remaining $(1-\sigma)\delta$ sellers flow into the posted-price market over a unit of time. This must equal the flow of purchases made by buyers that hit their deadline:

$$(1-\sigma)\delta = HF'(0). \tag{16}$$

[25] Very little changes if we allow the listing fee to differ across the two markets.

[26] While we refer to each seller as producing a single unit, one could also think of a seller offering multiple units so long as the production and listing costs scale up proportionately. The crucial elements are a negligible impact on the market and the correct stock and flow of units of the good available on the market.

At any moment, both markets will have a stock of active listings — sellers who are waiting for a buyer to make a purchase or for their auction to close. Let A denote the measure of auction sellers with active listings, and P denote the same for posted-price sellers. From the perspective of the individual auction seller, his auction will close at rate η; but with A sellers in the market at any instant, there will be ηA auctions that close over a unit of time. From the buyer's perspective, α auctions close over a unit of time; thus, these must equate in equilibrium:

$$\eta A = \alpha. \tag{17}$$

A similar condition applies to posted-price sellers. In aggregate, $HF'(0)$ purchases occur over a unit of time (sold to buyers who reach their deadline). From the individual seller's perspective, he can sell ζ units over one unit of time; collectively, these sellers expect to sell ζP units. In equilibrium, the expected sales must equal the expected purchases:

$$\zeta P = HF'(0). \tag{18}$$

4.4 Market Equilibrium Definition

With the addition of the seller's problem, we augment the equilibrium definition with three conditions. A *market steady-state equilibrium* consists of a buyer equilibrium as well as expected revenue $\theta^* \in \mathbb{R}^+$, expected profits $\Pi_a^* \in \mathbb{R}^+$ and $\Pi_p^* \in \mathbb{R}^+$, arrival rates $\alpha^* \in \mathbb{R}^+$ and $\zeta^* \in \mathbb{R}^+$, seller stocks $A^* \in \mathbb{R}^+$ and $P^* \in \mathbb{R}^+$, and fraction of sellers who enter the auction sector, $\sigma^* \in [0, 1]$, such that:

1. Expected revenue θ^* is computed using the bidding function $b^*(s)$ and distribution $F^*(s)$ derived from the buyer equilibrium, given α^*.

2. Prospective posted-price entrants earn zero expected profits: $\Pi_p^* = (1-\gamma)c$, given ζ^*.

3. Prospective auction entrants earn zero expected profits: $\Pi_a^* = (1-\gamma)c$ given $\alpha^* > 0$, or $\Pi_a^* \leq (1-\gamma)c$ if $\alpha^* = 0$.

4. α^*, ζ^*, σ^*, A^*, and P^* satisfy the steady state equations 15 through 18.

The first requirement simply imposes that buyers behave optimally as developed in the preceding section, given the endogenously-determined auction arrival rate. The fourth imposes the steady state conditions.

The second and third requirements impose zero profits for both types of sellers, which is necessary because of the large, unrestricted pool of potential entrants. If either market offered positive profits, additional sellers would be attracted to that market. This in turn reduces profits: more posted-price sellers P reduces the rate of selling ζ, and more auction sellers A increases the auction arrival rate and thus decreases expected revenue θ. Together, these two requirements also ensure that sellers are indifferent about which market they enter, thereby allowing them to randomize according to mixed strategy σ.

In the third requirement, we allow for the possibility that no auctions are offered, but this can only occur if the expected revenue from an auction would be weakly less than that of a posted-price listing. A similar possibility could be added to the posted-price market, but that market would never shut down in equilibrium. Due to the search friction, a fraction of buyers will inevitably reach their deadline; as a consequence, the posted-price market can always break even by reducing the stock of sellers waiting to serve these desperate buyers.

Since the posted price always exceeds the realized auction price, expected profits can only be equated if auction listings are sold more quickly. In other words, if both types of listings are offered in equilibrium, then $\zeta^* < \eta$.

4.5 Market Equilibrium Characterization

While the market equilibrium conditions simplify considerably, they still do not admit an analytic solution. Indeed, we now must numerically solve for α^* and λ^* simultaneously. However, all other equilibrium objects can be expressed in terms of these. The key additional equation comes from the third equilibrium requirement:

$$\theta^* = c + \frac{\ell + \rho c(1-\gamma)}{\eta\,(1 - e^{-\lambda^*})}. \tag{19}$$

This ensures that the expected revenue from each auction precisely covers the expected cost of listing and producing the good. The costs (on the right-hand side) are affected by λ because of the (small) chance that no bidders arrive, while expected revenue (on the left-hand side) is affected by both λ and α because of their influence on the bidding function and distribution of buyers. To compute θ, Equation 12 must be evaluated using $b(s)$ and $F(s)$ from the buyer equilibrium; the resulting equation is cumbersome and is reported in the proof of Proposition 3 in the Appendix.

At the same time, a buyer equilibrium requires that $\phi(\lambda^*) = 0$ from Equation 8; here, we note that this equation involves both λ and α. Equilibrium is attained when both Equation 8 and 19 simultaneously hold, which can only be solved numerically.

Once α^* and λ^* are found, the remaining equilibrium objects are easily solved as follows:

$$\Pi_a^* = (1-\gamma)c \tag{20}$$

$$\Pi_p^* = (1-\gamma)c \tag{21}$$

$$A^* = \frac{\alpha^*}{\eta} \tag{22}$$

$$P^* = \frac{(z-c)\left(\delta - \alpha^*\left(1-e^{-\lambda^*}\right)\right)}{\ell + \rho(1-\gamma)c} \tag{23}$$

$$\zeta^* = \frac{\ell + \rho(1-\gamma)c}{z-c} \tag{24}$$

$$\sigma^* = \frac{\alpha^*\left(1-e^{-\lambda^*}\right)}{\delta}. \tag{25}$$

It is readily apparent that $\sigma^* \geq 0$. To see that $\sigma^* < 1$, note that the equilibrium condition $\phi(\lambda^*) = 0$ requires that $\alpha\left(1-e^{-\lambda}\right) < \delta$. This also ensures that $P^* > 0$.

The following proposition demonstrates that these solutions are necessary for any equilibrium in which auctions actually take place.

Proposition 3. *A market equilibrium with active auctions ($\alpha^* > 0$) must satisfy $\phi(\lambda^*) = 0$, Equations 9 through 11, and Equations 19 through 25.*

The solution described in Proposition 3 can appropriately be called a *dispersed equilibrium*, to use the language of equilibrium search theory, because we observe the homogeneous good being sold at a variety of prices. Contrast this with a *degenerate equilibrium*, in which the good is always sold at the same price. In our context, this only happens if all goods are purchased via posted-price listings and no auctions are offered ($\alpha^* = \sigma^* = 0$). We can analytically solve for this degenerate equilibrium and for the conditions under which it exists, as described in the following proposition.

Proposition 4. *The degenerate market equilibrium, described by Equations 9 through 11 and Equations 21 through 25 with $\alpha^* = 0$ and $\lambda^* = \tau\delta T$, exists if and only if*

$$z - (z-\beta x)\left(1 - \frac{\delta\tau\left(\delta\tau + e^{-(\delta\tau+\rho)T}\left(-\delta\tau + \delta\rho\tau T + \rho^2 T\right)\right)}{(\delta\tau+\rho)^2\left(1-e^{-\tau\delta T}\right)}\right) \leq c + \frac{\ell + \rho c(1-\gamma)}{\eta\left(1-e^{-\tau\delta T}\right)}. \tag{26}$$

The left-hand side of Equation 26 is the expression for θ when $\alpha = 0$; it calculates the expected revenue that a seller would earn by offering an auction when no one else does. For this equilibrium to exist, the expected revenue must be lower than the expected cost of entering the market (the right-hand side of Eq. 26), so that not selling in the auction market is a best response. We can consider such a deviation because buyers still wait until their deadline before purchasing via the posted-price listing. Thus, there are $H^* = \delta T$

buyers in the market, uniformly distributed on $s \in [0, T]$, who are available as bidders in the measure-zero event that an auction occurs and would bid their reservation price $b(s) = \beta x + (z - \beta x)e^{-\rho s}$.

Equation 26 provides insight on when a degenerate solution will occur. For instance, on the right-hand side, one that see that a high production cost or listing cost can make the auction market unprofitable. The posted-price market can compensate for these costs by keeping a low stock of sellers so that the item is sold very quickly. Long delays before closing the auction (a small η) also increase the cost of the auction. On the left-hand side, δ and τ have the largest impact on expected revenue. With either a small flow of new buyers entering the market or a tiny fraction of them paying attention to a given auction, the number of participants per auction will be low. Without much competition in the second-price auction, expected revenue will be too low to cover expected costs.

Equilibrium search models frequently result in a degenerate equilibrium in which only one price is offered; in fact, the degenerate equilibrium often exists regardless of parameter values. This universal existence arises as a self-fulfilling prophecy: if buyers expect only one price to be offered, it is not worth incurring a search cost to look for other prices; but then sellers have no benefit from offering more than one price. Degenerate equilibria are less common in our auction environment because our buyers do not incur any cost to watch for auctions; even if no auctions are expected, buyers are still passively available should one occur.[27]

In fact, it appears that the degenerate equilibrium is mutually exclusive of the dispersed equilibrium, such that one exists if and only if the other does not. The complicated expression for θ^* in the dispersed equilibrium precludes an analytic proof of this conjecture, but we have observed this outcome in numerous calculations across a wide variety of parameters.

The underlying cause is quite intuitive: all else equal, more auctions lead to lower expected revenue per auction. Indeed, the comparative statics in Section 3.6 revealed that more auctions lead to fewer bidders per auction and cause buyers to lower their bids in anticipation of extra opportunities to win future auctions — both of which reduce expected revenue. That is to say, we conjecture that θ is a decreasing function of α (including the indirect effect of α on λ), and if so, uniqueness of equilibrium is assured. For example, if θ is less than the expected cost when $\alpha = 0$, then the degenerate equilibrium exists but the dispersed equilibrium cannot, since increasing α will further lower θ. On the other hand, if θ equals the expected cost for some $\alpha^* > 0$, then the dispersed equilibrium exists but the degenerate equilibrium cannot, since decreasing α to 0 will increase θ. Thus, the expected revenue from deviating from the no-auction equilibrium would strictly exceed the expected

[27] For example, eBay allows users to create "Searches you follow," in which the user enters search terms and is periodically informed of any newly listed items that match that search criteria.

cost.

To illustrate the outcome of this augmented model, we continue our calibration from Section 3.5. First, we can directly observe the average listing fee paid and the average time for which an auction is listed. We also observe the fraction of posted price listing that sell within the 30 day window of their listing, which is equivalent to ζ, the endogenous rate at which posted-price listings sell. We also note that α, which was calibrated in Table 1, is now endogenous. Here, we can use the equilibrium condition for α (Eq. 19) to determine what the underlying cost of production. The last two conditions are solved jointly to recover γ and c.

Table 3: Additional Calibration for Market Equilibrium

Fact	Observed Value	Theoretical Equivalent	Calibrated Parameter
Average listing fee paid	0.01	ℓ	$\ell = 0.01$
Average duration of an auction listing (months)	0.153	η	$\eta = 0.153$
Average % of posted-price listing sold in 30 days	1.4	$\frac{\ell+\rho(1-\gamma)c}{z-c}$	$\gamma = 0.25$
—	—	Eq. 19	$c = 0.882$

Under these parameters, a sale in the posted-price market generates an extra \$1.76 dollars on average, but this is offset in that the sale occurs after 35 periods on average. Note that we did not use the fraction of sellers in each format as a calibration target, providing us with another check for goodness of fit. Our model predicts that $\sigma^* = XX$ percent of new sellers will use the auction format, and that the stock of posted-price listings should be $A^*/P^* = 3$ times bigger than the stock of auction listings. In the data, these numbers are XX and XX, respectively.

4.6 Comparative Statics

We now present comparative statics for the market equilibrium. Here, the computation of θ^* prevents analytic determination of the sign of the comparative statics, but numeric evaluation remains consistent over a large space of parameter values. We highlight a few of these, noting that even if these phenomenons may not always occur, it is striking that they occur at all.

First, we note that for increases in τ under the market equilibrium, bidding respond opposite of what it does in the buyer equilibrium. The difference, of course, is that more auctions are offered in equilibrium when buyers are more attentive; this effect dominates the

Table 4: Comparative statics on key statistics: Market Equilibrium

		$\partial/\partial c$	$\partial/\partial\gamma$	$\partial/\partial\ell$	$\partial/\partial\tau$	$\partial/\partial T$	$\partial/\partial\rho$
Auction Rate	α^*	$-$	$+$	$-$	$+$	$-$	$-$
Participants per Auction	λ^*	$+$	$-$	$+$	$+$	$+$	$+$
% Buying via Posted Price	$F'(0)H^*$	$+$	$-$	$+$	$-$	$+$	$+$
Stock of Posted Price Sellers	P^*	$+$	$+$	$+$	$-$	$-$	$+$
Lowest Bid	$b^*(T)$	$+$	$-$	$+$	$-$	$-$	$-$
Expected Revenue	θ^*	$+$	$-$	$+$	$-$	$-$	$+$

Note: Reported signs are for numeric computations on the example.

increased number of participants per auction. Similarly in the market equilibrium, increases in ρ will now affect the distribution of buyers because fewer auctions are offered; this leads to higher expected revenue despite lower bids. Changes in T behave similarly under either equilibrium definition.

For c, it is remarkable that even though increased production costs do not raise the retail price (by assumption), they still affect auctions in the distribution of buyers and their bids. Higher costs will shrink the margins in both markets, which the auction market responds to by reducing its flow of sellers. Fewer auctions necessarily mean that more buyers reach their deadline; and this increased demand for posted-price listings more than compensates for the smaller margin. That is, a larger stock of posted-price sellers is needed to return to normal profits. Also, because there are fewer auctions, there is less value in waiting for future auctions. This drives up bidders' reservation prices, but not enough to prevent a smaller flow of auction sellers.

A higher listing fee, ℓ, has a similar effect, but this is surprising because the listing fee applies to both markets and will be paid more often by posted-price sellers (who have to relist their good more times). Yet again, this drives sellers away from the auction market, and any reduction in the rate of auctions will increase bids.

This subtle response illustrates a potential hazard in auction design if buyer valuations are not fundamental but rather the endogenous results of deeper factors. A seemingly neutral change in the auction listing fee not only alters which market sellers use, but also warps the distribution of buyer valuations.

4.7 Reserve Prices

To this point, we have assumed that auction sellers always set their reserve price equal to $b(T)$, the lowest bid any buyer might make in equilibrium. There is clearly no incentive to reduce the reserve price below that point: doing so would not bring in any additional bidders, but would decrease revenue in those instances where only one bidder participates.

Now consider a seller who considers raising the reserve price to $\tilde{R} > b(T)$, taking the behavior of all others in the market as given. This will only affect the seller when a single bidder arrives or the second highest bid is less than $\tilde{R}$; with this higher reserve price, the seller closes the auction without sale in these situations and re-lists the good, a strategy which has a present discounted value of Π_a. Of course, the seller gives up the immediate revenue and completion cost, which is no more than $\tilde{R} - \gamma c$.

Since $\Pi_a = (1-\gamma)c$ in equilibrium, deviating to the reserve price $\tilde{R}$ is unprofitable if $\tilde{R} - \gamma c \leq (1-\gamma)c$, or rearranged, $\tilde{R} \leq c$. In words, the optimal seller reserve price should equal the total cost of production — a common result for auction models. Thus, in our context, $b(T)$ is the optimal seller reserve price so long as $b^*(T) \geq c$. Otherwise, one would set the reserve price to c.

The latter case can still be solved using our model by endogenizing the buyer deadline, T. For instance, suppose that buyers who enter six months before their deadline are only willing to bid below the cost of production. By raising the reserve price, these bidders are effectively excluded from all auctions; it is as if they do not exist. They only begin to participate once they reach time S such that $b^*(S) = c$. In other words, it is as if all buyers enter the market with S time until their deadline. To express this in terms of our model, we make T endogenous, requiring $b^*(T^*) = c$ in equilibrium. All else will proceed as before.

Of course, even with optimal reserve prices, the entry and exit of sellers will ensure that expected profits from entering the market are zero. Any gains from raising the reserve price are dissipated as more auctions are listed. To consider the absence of this competitive response, consider if one seller had monopoly control of both markets. The optimal choice would be to shut down the auction market, forcing all buyers to purchase at the highest price z. Competition among sellers, however, undermines this degenerate equilibrium (at least when Equation 26 does not hold); sellers see auctions as an opportunity to sell their product faster, even if at a slightly lower price and to the detriment of other sellers.

5 Empirical Analysis of eBay Data

[COMING SOON]

6 Conclusion

This work reexamines the auction environment as a venue for selling retail goods. Our analysis leverages methods frequently used in search theory, which provide analytic tractability and plausibly match the auction setting. Standard auction theory ascribes all variation in bids as generated by exogenous differences in valuations; but this seems less compelling when there is a readily-available outside option for the same item. In our model, all buyers enjoy the same eventual utility from the good, but endogenously differ in how soon they must acquire it. This produces an increasing and accelerating path for an individual's bidding over time, and a rich continuous distribution of closing prices across auctions.

The model makes heavy use of homogeneity, with buyers who only differ in how long they have been in the market and sellers who only differ because of their mixed strategy of which market to enter. This homogeneity makes the solution more tractable, but also serves to set the model in sharp relief to those with exogenous valuations. Indeed, recovering the distribution of valuations is a key focus of empirical work in the auction literature, yet it is plausible that these distributions are themselves a product of more fundamental factors, such as impending deadlines. If so, one should not expect the distribution of valuations to be invariant to policy interventions or changes in the auction design. As we have shown, even something as simple as increasing the listing fee — for both auctions and posted-price listings — can change buyer valuations.

At the same time, one could add realism by allowing heterogeneity among buyers or sellers. For instance, buyers could enter with differing deadlines or differing final values, or sellers could differ in their costs of production or discount rate. These extensions would complicate our steady state conditions, but the mechanisms underlying our solution would still govern the final solution.

A Proofs

Derivation of Bellman Equations. Each of the continuous-time Bellman equations (Eq. 2, 13, and 14) in the model can be derived from a discrete-time formulation as follows. First, consider the expected profit of a seller in the auction market, Π_a. Let Δ be the length of a period of time, which we assume to be sufficiently short such that $\eta\Delta < 1$; this can then be interpreted as the probability of the auction closing during that period of time. The discrete time Bellman equation is thus:

$$\Pi_a = -\ell\Delta + \frac{1}{1+\rho\Delta}\left(\eta\Delta\left(1-e^{-\lambda}\right)(\theta-\gamma c) + \left(1-\eta\Delta\left(1-e^{-\lambda}\right)\right)\Pi_a\right). \tag{27}$$

The term $\ell\Delta$ is the listing fee incurred during the period of time. The term in parenthesis computes the expected outcome in the next period of time: either the auction closes with at least one bidder, earning $\theta - \gamma c$, or it does not close or attracts no bidders, so the seller enters the next period with the same expected payoffs as the current period. These future payoffs are discounted by the factor $1/(1+\rho\Delta)$.

By moving $\Pi_a/(1+\rho\Delta)$ to the left-hand side, then dividing by Δ, this becomes:

$$\frac{\rho}{1+\rho\Delta}\Pi_a = -\ell + \frac{\zeta\left(1-e^{-\lambda}\right)}{1+\rho\Delta}\left(\theta-\gamma c-\Pi_a\right),$$

and taking the limit as $\Delta \to 0$, we obtain Eq. 13.

The expected profit for posted-price sellers is derived similarly. Again, we assume that a period is short enough that $\zeta\Delta < 1$.

$$\Pi_p = -\ell\Delta + \frac{1}{1+\rho\Delta}\left(\zeta\Delta(z-\gamma c) + (1-\zeta\Delta)\Pi_p\right). \tag{28}$$

Like auction sellers, posted-price sellers incur the listing fee $\ell\Delta$. With probability $\zeta\Delta$, they encounter a buyer in the next period and earn $z-\gamma c$; otherwise they continue waiting. This rearranges as:

$$\frac{\rho}{1+\rho\Delta}\Pi_p = -\ell + \frac{\zeta}{1+\rho\Delta}\left(z-\gamma c-\Pi_p\right),$$

and taking the limit as $\Delta \to 0$, we obtain Eq. 14.

The derivation for the buyer's expected utility is similar, only with more sources of uncertainty if an auction occurs. Let the period length Δ be sufficiently short that $\alpha\tau\Delta < 1$. This can then be interpreted as the probability that an auction occurs during the unit of time. A buyer's expected utility in state s can be expressed as follows:

$$\begin{aligned}V(s) = \frac{1}{1+\rho\Delta}\Bigg(&\left(1 - \tau\alpha\Delta \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}(1-F(s))^n\right) V(s-\Delta) \\ &+ \tau\alpha\Delta\left(\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}(1-F(s))^n \left(\beta + (1-\beta)e^{-\rho s}\right) x \right. \\ &\left. - e^{-\lambda}b(T) - \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \int_s^T b(t)n(1-F(t))^{n-1}F'(t)dt\right)\Bigg).\end{aligned} \tag{29}$$

On the left-hand side, all utility is discounted by factor $1/(1+\rho\Delta)$, meaning that the buyer does not receive any utility during the current period. By the next period, one of two outcomes could occur: either the buyer wins an auction and exits (second and third lines of Eq. 29), or he continues his search (first line, due to losing or not participating).

Specifically, the second line computes the probability of the individual participating in an auction ($\tau\alpha\Delta$) and winning (the first two terms of the summation), times the utility enjoyed from winning ($\left(\beta + (1-\beta)e^{-\rho s}\right) x$). The third line compute the expected second-highest bid times the probability of winning and thus paying it. The first line considers when the buyer does not win or does not participate (the probability in parentheses), in which case the buyer will continue waiting for future auction opportunities. Yet, he will do so with less time remaining before his deadline, reflected in his state changing to $s - \Delta$.

To transform this to a continuous-time Bellman equation, we first multiply both sides by $(1+\rho\Delta)/\Delta$, then subtract $V(s)/\Delta$ from both sides, obtaining:

$$\begin{aligned}\rho V(s) = \frac{V(s-\Delta) - V(s)}{\Delta} + \tau\alpha&\left(\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}(1-F(s))^n \left(\left(\beta + (1-\beta)e^{-\rho s}\right) x - V(s-\Delta)\right)\right. \\ &\left. - e^{-\lambda}b(T) - \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \int_s^T b(t)n(1-F(t))^{n-1}F'(t)dt\right).\end{aligned}$$

Then, by letting $\Delta \to 0$, we obtain the continuous-time Bellman Eq. 2.

□

Proof of Proposition 1. First, we note that the infinite sums in equations 2 and 4 can be readily simplified. In the case of the latter, it becomes:

$$F''(s) = \alpha\tau F'(s)e^{-\lambda F(s)}. \tag{30}$$

This differential equation has the following unique solution, with two constants of integration

k and m.

$$F(s) = \frac{1}{\lambda} \ln \left(\frac{\alpha\tau - e^{\lambda k(s+m)}}{\lambda k} \right). \tag{31}$$

The constants are determined by our two boundary conditions. Applying Eq. 6, we obtain $m = \frac{1}{\lambda k} \ln\left(\alpha\tau - \lambda k e^{\lambda}\right) - T$. By substituting this into Eq. 31, one obtains:

$$F(s) = \frac{1}{\lambda} \ln \left(\frac{\alpha\tau - e^{\lambda k(s-T)}\left(\alpha\tau - \lambda k e^{\lambda}\right)}{\lambda k} \right). \tag{32}$$

The other boundary condition, Eq. 5, requires that k to satisfy:

$$\alpha\tau\left(1 - e^{-\lambda T k}\right) - \lambda k\left(1 - e^{\lambda - \lambda T k}\right) = 0. \tag{33}$$

From Eq. 7, we know that $H = \delta/F'(T)$, and using the solution for F in Eq. 32, this yields $H = \delta\lambda/\left(\lambda k - \alpha e^{-\lambda}\tau\right)$. We then substitute this into the fourth equilibrium requirement, $\lambda = \tau H$, and solve for k to obtain:

$$k = \frac{\tau}{\lambda}\left(\delta + \alpha e^{-\lambda}\right). \tag{34}$$

When we substitute this for k in Eq. 32, we obtain the equilibrium solution for F^* depicted in Eq. 9. Also, when Eq. 34 is used to replace k in the boundary condition in Eq. 33, we obtain the formula ϕ (Eq. 8) which implicitly solves for λ^*.

We now show that a solution always exists to $\phi(\lambda^*) = 0$ and is unique. Note that as $\lambda \to +\infty$, $\phi(\lambda) \to +\infty$. Also, $\phi(0) = -\delta\left(1 - e^{-\tau(\alpha+\delta)T}\right) < 0$. Since ϕ is a continuous function, there exists a $\lambda^* \in (0, +\infty)$ such that $\phi(\lambda^*) = 0$.

We next turn to uniqueness. The derivative of ϕ w.r.t. λ is always positive:

$$\phi'(\lambda) = \alpha e^{-\lambda} + \delta(e^{\lambda} + \alpha\tau T)e^{-\tau(\alpha e^{-\lambda}+\delta)T} > 0.$$

Thus, as an increasing function, $\phi(\lambda)$ crosses zero only one time, at λ^*.

We finally turn to the solution for the bidding function. Again, we start by simplifying the infinite sums in Eq. 2. The first sum is similar to that in Eq. 4. For the second, we first change the order of operation, to evaluate the sum inside the integral. This is permissible by the monotone convergence theorem, because $F(s)$ is monotone and $\sum \frac{e^{-\lambda}\lambda^n}{n!} b(t)n(1 -$

$F(t))^{n-1}$ converges uniformly on $t \in [0, T]$. After evaluating both sums, we obtain:

$$\rho V(s) = -V'(s) + \alpha\tau \Bigg(e^{-\lambda F(s)} \left(\left(\beta + (1-\beta)e^{-\rho s}\right) x - V(s)\right) \\ -e^{-\lambda} b(T) - \int_s^T \lambda e^{-\lambda F(t)} b(t) F'(t) dt \Bigg).$$

Next, by taking the derivative of $b(s) = \left(\beta + (1-\beta)e^{-\rho s}\right) x - V(s)$ (Eq. 1), we obtain $b'(s) = -\rho(1-\beta)xe^{-\rho s} - V'(s)$. We use these two equations to substitute for $V(s)$ and $V'(s)$, obtaining:

$$(\rho + \alpha\tau e^{-\lambda F(s)})b(s) + b'(s) = \rho\beta x + \alpha\tau \left(e^{-\lambda} b(T) + \int_s^T \lambda e^{-\lambda F(t)} b(t) F'(t) dt \right). \tag{35}$$

This equation holds only if its derivative with respect to s also holds, which is:

$$(\rho + \alpha\tau e^{-\lambda F(s)})b'(s) + b''(s) = 0.$$

After substituting for $F(s)$ solved above, this differential equation has the following unique solution, with two constants of integration a_1 and a_2:

$$b(s) = a_1 \cdot \left(\frac{\delta e^{\lambda^* - \tau T\left(\delta + \alpha e^{-\lambda^*}\right)}}{\rho} + \frac{\alpha e^{-\tau s\left(\delta + \alpha e^{-\lambda^*}\right)}}{\rho + \tau\left(\delta + \alpha e^{-\lambda^*}\right)} \right) e^{-s\rho} + a_2. \tag{36}$$

This solves the differential equation; but to satisfy Eq. 35, a particular constant of integration must be used. We substitute for $b(s)$ in Eq. 35 using Eq. 36, and solve for a_2. This can be done at any $s \in [0, T]$ with equivalent results, but is least complicated at $s = T$ since the integral disappears: $(\rho + \alpha\tau e^{-\lambda F(T)})b(T) + b'(T) = \rho\beta x + \alpha\tau e^{-\lambda} b(T)$. After substituting $b(T)$, $b'(T)$, and $F(T)$, solving for a_2 yields:

$$a_2 = \beta x + a_1 \frac{\alpha\tau\left(\delta + \alpha e^{-\lambda^*}\right)}{\rho\left(\rho + \delta\tau + \alpha\tau e^{-\lambda^*}\right)} e^{-\rho T - \tau T\left(\delta + \alpha e^{-\lambda^*}\right)}. \tag{37}$$

The other constant of integration is determined by boundary condition Eq. 3. If we translate this in terms of $b(s)$ as we did for the interior of the Bellman Equation, we get $b(0) = z$. We then substitute for $b(0)$ using Eq. 36 evaluated at 0, and substitute for a_2 using Eq. 37, then solve for a_1:

$$a_1 = \frac{\rho(z - \beta x)\left(\rho + \delta\tau + \alpha\tau e^{-\lambda^*}\right) e^{\tau T\left(\delta + \alpha e^{-\lambda^*}\right)}}{\delta\rho e^{\lambda^*} + \tau\left(\delta + \alpha e^{-\lambda^*}\right)\left(\delta e^{\lambda^*} + \alpha e^{-\rho T}\right) + \alpha\rho e^{\tau T\left(\delta + \alpha e^{-\lambda^*}\right)}}.$$

If the solutions for a_1 and a_2 are both substituted into Eq. 36, one obtains Eq. 11 with minor simplification. $\square$

Proof of Proposition 2. The first derivative of $b^*(s)$ is:

$$b'(s) = -\frac{\rho(z-\beta x)\left(\rho+\delta\tau+\alpha\tau e^{-\lambda^*}\right)\left(\delta e^{\lambda^*}+\alpha e^{\tau(T-s)\left(\delta+\alpha e^{-\lambda^*}\right)}\right)}{\delta\rho e^{\lambda^*}+\tau\left(\delta+\alpha e^{-\lambda^*}\right)\left(\delta e^{\lambda^*}+\alpha e^{-\rho T}\right)+\alpha\rho e^{\tau T\left(\delta+\alpha e^{-\lambda^*}\right)}}.$$

Note that each of the parenthetical terms is strictly positive, thus the negative in front ensures that the derivative is negative.

The second derivative is:

$$b''(s) = \frac{\rho(z-\beta x)\left(\rho+\delta\tau+\alpha\tau e^{-\lambda^*}\right)\left(\delta\rho e^{\lambda^*}+\alpha\left(\rho+\delta\tau+\alpha\tau e^{-\lambda^*}\right)e^{\tau(T-s)\left(\delta+\alpha e^{-\lambda^*}\right)}\right)}{\delta\rho e^{\lambda^*}+\tau\left(\delta+\alpha e^{-\lambda^*}\right)\left(\delta e^{\lambda^*}+\alpha e^{-\rho T}\right)+\alpha\rho e^{\tau T\left(\delta+\alpha e^{-\lambda^*}\right)}}e^{-s\rho}.$$

Again, each parenthetical term is positive. Hence $b''(s) > 0$. $\square$

Proof of Proposition 3. By Proposition 1, Eqs. 9 through 11 and $\phi(\lambda^*) = 0$ must be satisfied in order to be a buyer equilibrium, as required in the first condition.

The solutions to A^* and σ^* are simply restatements of Eq. 17 and 15, respectively.

The profits stated in Eqs. 20 and 21 are required by the third and second equilibrium conditions, respectively. From Eq. 14, profit solves as: $\Pi_p = \frac{\eta(z-\gamma c)-\ell}{\rho+\zeta}$, so for this to equal $(1-\gamma)c$, we require $\zeta^* = \frac{\ell+\rho c(1-\gamma)}{z-c}$. With this, Eq. 16 readily yields P^* as listed in Eq. 23.

The only remaining element is regarding expected auction profit. Eq. 13 solves as: $\Pi_a = \frac{\eta\left(1-e^{-\lambda}\right)(\theta-c\gamma)-\ell}{\eta\left(1-e^{-\lambda}\right)+\rho}$. By setting this equal to $(1-\gamma)c$ and solving for θ, we obtain Eq. 19.

To evaluate the integrals in θ, we first note that by interchanging the sum and integral and evaluating the sum, expected revenue simplifies to:

$$\theta = \frac{\lambda}{1-e^{-\lambda}}\left(e^{-\lambda}b(T)+\lambda\int_0^T b(s)F(s)F'(s)ds\right). \tag{38}$$

After substituting for $b(s)$ and $F(s)$ from the buyer equilibrium, this evaluates to:

$$\theta = \frac{1}{1-e^{-\lambda}} \cdot \Bigg(z - \beta x e^{-\lambda} + \frac{z-\beta x}{(\rho+\kappa\tau)\left(\rho\delta + \tau(\kappa-\alpha)\left(\delta + \alpha e^{-\lambda-\rho T}\right)\right)} \cdot$$
$$\Bigg((\alpha-\kappa)e^{-\lambda-\rho T}\left(\kappa\tau(\kappa\tau-\lambda\rho)-\lambda\rho^2\right) - \delta\rho(2\kappa\tau+\rho)$$
$$+\kappa\rho\tau\left(\delta\Psi\left(1-\frac{\kappa}{\alpha}\right) + (\alpha-\kappa)e^{-\lambda-\rho T}\Psi\left(1-\frac{\kappa e^{\lambda}}{\alpha}\right)\right)\Bigg)\Bigg),$$

where $\kappa \equiv \delta + \alpha e^{-\lambda}$ and $\Psi(q)$ is Gauss's hypergeometric function with parameters $a = 1$, $b = -1 - (\rho/\tau\kappa)$, $c = -\rho/\tau\kappa$, evaluated at q. Under these parameters, the hypergeometric function is equivalent to the integral:

$$\Psi(q) \equiv -\left(1 + \frac{\rho}{\tau\kappa}\right)\int_0^1 \frac{t^{-2-\frac{\rho}{\tau\kappa}}}{1-qt}dt.$$

While not analytically solvable for these parameters, Ψ is readily computed numerically. □

Market Efficiency. As we consider the contribution to total welfare by this market, the degenerate equilibrium provides a good benchmark. For exposition purposes, first consider welfare taking the rate of auctions α as given, as in the buyers equilibrium (that is, ignore Eq. 19). If no auctions were available, all buyers would wait until the last minute then purchase at the posted price. The net gain to society over a unit of time would be $\delta(x - c)$, as δ buyers are served, δ sellers incur completion costs γc, and δ other sellers incur initial production $(1 - \gamma)c$ as they enter the market.

An additional concern is the listing fees these sellers incur. If we include the market host (eBay) in total welfare, then listing fees are merely a transfer between sellers and the host. Yet the listing fee reflects a sort of inefficiency in the market caused by the matching friction. For instance, with perfect coordination, a buyer and seller could instantly be matched on entry so that no stock P of posted-price listings needed to be maintained. This does not occur because of the random arrival and matching of buyers to sellers in P. Through their excessive entry, sellers compete away their profits into listing fees. In the degenerate case, total listing fees would be ℓP.

Now consider the dispersed equilibrium. For sellers, total costs incurred over a unit of time are $\Gamma \equiv \ell(A + P) + \delta c$, which includes the listing cost for the stock of uncompleted auctions. One can substitute for the equilibrium stock of both sellers, but the intuition can be seen without doing so. Because posted-price sellers require more time to sell than auctions do, any change in the market that decreases P by 1 will increase A by less than 1, though still serving δ buyers per unit of time. In other words, the market is more efficient as it serves the same population while reduces the volume of listings.

Buyers in the dispersed equilibrium will experience different utility depending on when they buy: obtaining the good with s time remaining provides $\left(\beta + (1-\beta)e^{-\rho s}\right) x$ utils. Because of this, early acquisition is inefficient; the production occurs immediately but some of the consumption is discounted. Of course, in a given auction, the highest valuation bidder will always win, as in any sealed-bid second price auction. However, changes in the auction environment will alter the distribution and number of bidders in each auction, so we must ask how much utility the average winner receives from the good, prior to paying for it.

The density of the highest bidder in a given auction (conditional on having any bidders show up) is given by:

$$\begin{aligned} F_1'(s) &= \frac{1}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} n(1-F(t))^n F'(t)dt \\ &= \frac{\lambda e^{-\lambda F(s)} F'(s)}{1-e^{-\lambda}}. \end{aligned}$$

Auctions occur at rate α per unit of time, and attract at least one buyer with probability $1-e^{-\lambda}$. In addition, $H \cdot F'(0)$ buyers will purchase at the posted price per unit of time, enjoying the full utility x. Summing across all buyers, the total utility U generated by the entire market over each unit of time is:

$$U \equiv x\left(\delta - \alpha\left(1-e^{-\lambda}\right)\right) + x\alpha \int_0^T \left(\beta + (1-\beta)e^{-\rho s}\right) \lambda e^{-\lambda F(s)} F'(s) ds. \tag{39}$$

The integral in Eq. 39 evaluates with similar complexity to expected revenue, precluding analytic analysis. However, with significant algebraic manipulation, one can show that for the cumulative distribution $F_1(s)$, $\frac{\partial F_1(s)}{\partial \alpha} \leq 0$ for all $s \in [0,T]$, meaning that any increase in the number of auctions will decrease the fraction of winners who are close to their deadline. That is, the distribution with fewer auctions first-order stochastically dominates the distribution with more auctions. Since the utility enjoyed at a given s is unaffected by α, more auctions unambiguously increases average consumer utility.

Intuitively, this happens because frequent auctions provide more opportunities to win, so bidders exit earlier in their search spell; in addition, it leads to fewer participants per auction. While both of these are beneficial to the consumer, they ensure that the average winner is further from his deadline. Indeed, by this logic, the degenerate equilibrium produces the most utility from the goods by ensuring that they are fully enjoyed at the time of purchase.

The total welfare added through this market is $W \equiv U - \Gamma$. The two terms move in opposite directions, with more auctions reducing costs but increasing early acquisition; the net effect can only be compared in numeric examples. However, across a wide variety of parameterizations, the cost effect consistently dominates the utility effect. That is, more auctions are strictly beneficial because they decrease listing costs by more than they reduce average utility. This remains true even when the listing fee is quite low or the discount rate is quite high. Indeed, in every computation we have performed where a dispersed equilibrium existed, total welfare was greater than if it were forced into a degenerate market (by outlawing auctions, for instance).

The preceding analysis has focused on how the auction rate affects welfare. In the market equilibrium, this rate is endogenously determined. In numerically evaluating the welfare consequences of the fundamental parameters, though, these exclusively depend on how that parameter affects the equilibrium auction rate. For instance, a larger listing fee (ℓ) will reduce the auction rate and total welfare, while deferring more costs the time of sale (γ) will increase the auction rate and total welfare. In short, total welfare seems to track perfectly with the endogenous auction rate. □

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