Is Human Capital Under-Produced?

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Abstract

If there are substantial positive externalities in the production of human capital, it is under-produced in a competitive equilibrium. We calibrate a growth model with leisure and with two sectors – human capital production and goods production – to estimate the size of such externalities. Then, we study the effects of subsidies on human capital production, as well as taxes, on welfare. The results suggest that increased subsidies or decreased income taxes or both, financed by increased consumption taxes, would increase the welfare in the United States.

\textit{Keywords:} Optimal Fiscal Policy, Human Capital Externalities

\textit{JEL Classifications:} H21, H23, O41

* The views expressed in this paper are those of the authors and do not necessarily represent those of the IMF or IMF policy.

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1. Introduction

Numerous studies report large human capital externalities, either in goods production (Moretti, 2004; Liu, 2006; Dalmazzo and de Blasio, 2007; Muravyev, 2008) or in human capital production (Borjas, 1992, 1995; Choi, 2011). That is, an agent’s human capital production, such as education, training, and research and development (R&D), not only increases his own productivity, but also tends to increase the productivity of other agents. If there are such externalities, then there is a welfare loss in a competitive equilibrium since agents do not consider externalities in their individual decisions. This situation can be “corrected” by government subsidies on human capital production. Indeed, many governments allocate a great deal of resources to subsidize education, training, and R&D. For example, the 1960-2005 average of the ratio of government expenditure on education to total expenditure on education is 75 percent in the United States (Choi, 2011). In addition, the U.S. subsidy rate on private R&D investments is 20 percent (Hall and Van Reenen, 2000).

What is the optimal subsidy rate on human capital production, reflecting the size of the estimated human capital externalities? Traditional approaches to the optimal fiscal policy (e.g., Rebelo, 1991) have attempted to choose tax rates on various tax bases to maximize the welfare level. However, most of them assume no externalities. The goal of this study is to consider the subsidies to human capital production as a fiscal tool, along with labor income tax, capital income tax, and consumption tax, to solve for the optimal fiscal policy that reflects the calibrated size of human capital externalities. The optimal fiscal policy implied by the model is compared to the status quo in order to determine whether the U.S. government should increase or decrease the current subsidy rate. The results suggest that the current subsidies are smaller than the socially optimal level.

This paper’s major contributions are two-fold. First, studies on the optimal investment in human capital are usually based on the direct micro-level estimation of returns

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3While this paper focuses on quantifying externalities, some studies ask what causes these externalities. Acemoglu (1996) suggests that high-skilled workers convince companies to invest more in physical capital which benefits low-skilled workers. Jovanovic and Rob (1989) suggest that an increased level of knowledge makes the discovery of new knowledge more effective.
on schooling. (See, e.g., Heckman, et al. (2008).) Our approach is unique – It is based on macro-level calibrations. Second, there have been discussions on optimal fiscal policy, but this paper is among the first to consider the subsidies on human capital production as a tool of the fiscal policy. Exceptions include Gómez (2003, 2005) providing qualitative implications on the optimal fiscal policy in an economy with human capital externalities. However, this paper focuses on quantitative implications. Schindler and Yang (2010) theoretically analyze the effects of both subsidies to human capital production and taxes on physical capital in a theoretical model. However, the analysis remains qualitative while the model does not consider distortions caused by positive externalities to education as described in this paper.

This paper is organized as follows. In the rest of this section, we discuss the literature. In Section 2, we outline and solve for a model with human capital externalities. Further, we calibrate the model based on the U.S. data. In Section 3, we obtain the social planner’s solution. We show that taxes and subsides cannot be used to achieve this solution in a competitive equilibrium. Hence, we proceed to solve for the second-best solution. Section 4 concludes.

1.1. Related Literature

While Choi (2011) is a positive study providing an estimate of human capital externalities, this paper is a normative study discussing how to improve the welfare level given an estimate of human capital externalities. In addition, this paper considers a more realistic model including leisure. This paper also complements Heckman, et al. (2008) who use micro-level estimates to argue that the internal rate of return on human capital production is higher than the interest rate. This implies that human capital is under-produced. While their estimation assumes that all economic growth is due to human capital production and that education is the only form of human capital production, this paper’s theoretical model is able to introduce exogenous productivity growth, as well as

\footnote{In this paper, externalities are captured through the average human capital stock while in Gómez (2005), only the fraction of time devoted to human capital production enters.}
other forms of human capital production such as training and R&D.\(^5\) Another paper closely related to this paper is Gómez and Sequeira (2012). While we focus on the welfare issue in the existence of human capital externalities, Gómez and Sequeira (2012) do so in the existence of R&D spillovers.

One of the first papers to study the size of human capital externalities is Rauch (1993). Rauch (1993) used Mincerian wage regressions on metropolitan statistical areas to confirm the existence of benefits to human capital not reflected in private wages. Other papers, with statistical refinements, find similar results with Mincerian regressions. Rudd (2000), for example, uses more consistent state level panel data to account for area fixed effects. However, Ciccone and Peri (2006) point out that Mincerian regressions may suggest positive externalities even in cases where the private and social marginal benefits are equal when different levels of human capital are not perfect substitutes for each other. Using the constant composition approach to deal with this issue, Garcia-Fontes and Hidalgo (2009) still find positive externalities to human capital production. This paper provides a different approach to the measurement of human capital externalities using macro-level data which circumvents some issues discussed in the micro-level literature.

2. The Model

2.1. Model Description

The closed economy has an infinitely-lived representative consumer who also serves as a producer. Goods production follows a constant-returns-to-scale Cobb-Douglas function with two inputs, the services of physical capital and human capital. That is,

\[
Y(t) = K(t)^\alpha [A(t)[1 - u(t) - l(t)] h(t) L(t)]^{1-\alpha}
\]

for \(0 < \alpha < 1\) where \(Y(t)\) is the output (physical goods) produced, \(K(t)\) is the physical capital stock, \(A(t)\) is the labor-augmenting productivity which grows exogenously at \(g_A \geq 0\), \(u(t)\) and \(l(t)\) are fractions of human capital stock devoted to human capital production and leisure, respectively (making \(1 - u(t) - l(t)\) the fraction devoted to

\(^5\)Advantages of this paper’s macro calibration approach are discussed in Choi (2011).
goods production), $h(t)$ is the per-capita human capital stock, and $L(t)$ is the population which grows exogenously at $g_L \geq 0$. The path of $\{h(t)\}$ is taken as exogenous by the consumer. The law of motion of physical capital is $\dot{K}(t) = I(t) - \delta_K K(t)$, where $I(t)$ is the investment in physical capital accumulation, and $0 \leq \delta_K < 1$ is a depreciation rate of physical capital. All $Y(t)$, $K(t)$, and $I(t)$ are in units of physical goods. The law of motion of per-capita human capital is

\begin{equation}
\dot{h}(t) = B(t) [u(t) h(t)]^\phi \bar{h}(t)^\theta - (\delta_h + g_L) h(t),
\end{equation}

for $0 < \phi < 1$, $0 < \theta < 1$, $\phi + \theta \leq 1$ (which enables the model to have a balanced growth path) and $0 \leq \delta_h < 1$. Here, $B(t)$ is the productivity of human capital production which grows exogenously at $g_B \geq 0$. Also, $\bar{h}(t)$ is the average human capital stock in the economy. The first term of the right-hand side is the individual human capital production function, Cobb-Douglas with two inputs: (i) services of human capital and (ii) learning externalities reflected in the average human capital stock. Each period, a fraction $\delta_h$ of the human capital stock depreciates. Since $h(t)$ is in a per-capita term, its accumulation is adjusted by $g_L$.

We discuss three features of the model. First, the model is semi-endogenous, featuring two engines: human capital accumulation and exogenous productivity growth. An alternative specification would be to consider an exogenous growth model, in which $A(t)$ is the sole engine of growth and $h(t)$ determines the relative income level.\(^6\) Another specification is to assume that the sole source of growth is due to human capital accumulation, as in Heckman, et al. (2008) and Lucas and Moll (2011). An advantage of our model is that it allows both engines of growth.

Second, the model disregards physical inputs in human capital production. Considering physical inputs in (2) complicates the analysis without changing the calibration results substantially.\(^7\)

Third, the productivity in human capital production, $B(t)$, is allowed to grow. By

\(^6\)As it becomes clear later, this alternative specification would require additional data observations for calibration. The differences in human capital production activities across economies or over periods can be useful.

\(^7\)Choi (2011) includes a comparison of results with and without physical inputs in a related model.
construction, this leaves less room for learning externalities since $B(t)$ growth becomes an additional source of human capital accumulation. As a result, this set-up provides a more conservative estimate of learning externalities.

The resource constraint is

$$Y(t) = I(t) + C(t) + \tau_C C(t) + \tau_K r(t) K(t)$$

$$+ \tau_L w(t) A(t) [1 - u(t) - l(t)] h(t) L(t) - sw(t) A(t) u(t) h(t) L(t),$$

where $C(t)$ is the physical goods consumed, and $\tau_C$, $\tau_K$ and $\tau_L$ are constant tax rates on consumption, physical capital income, and human capital income (or labor income) in goods production. In addition,

$$r(t) \equiv \alpha K(t)^{\alpha-1} [A(t) [1 - u(t) - l(t)] h(t) L(t)]^{1-\alpha}$$

is the interest rate before taxes are imposed. Similarly,

$$w(t) \equiv (1 - \alpha) K(t)^{\alpha} [A(t) [1 - u(t) - l(t)] h(t) L(t)]^{-\alpha}$$

is the wage for an effective human capital unit, augmented by exogenous productivity $A(t)$, before taxes are imposed, in goods production. Thus, the wage for one unit of human capital becomes $w(t) A(t)$. We define $w(t)$ in this way to keep $w(t)$ constant on the balanced growth path. Finally, $sw(t)$, where $s$ is a constant subsidy rate, is a subsidy provided per effective human capital unit in human capital production. As the tax on human capital income is $\tau_L w(t)$ units of physical goods per effective human capital stock devoted to physical goods production, the subsidy is $sw(t)$ units of physical goods per effective human capital stock devoted to human capital production.

Using (1) and (5), we can rearrange the last term of (3), $sw(t) A(t) u(t) h(t) L(t)$, as $\frac{(1 - \alpha) su(t)}{1 - u(t) - l(t)} Y(t)$. In this term, $(1 - \alpha) Y(t)$ is the human capital income in goods production. Since the fractions of human capital stocks devoted to goods production and human capital production are $1 - u(t) - l(t)$ and $u(t)$, respectively, $\frac{(1 - \alpha) u(t)}{1 - u(t) - l(t)} Y(t)$ becomes the human capital income in human capital production. The subsidy is a fraction $s$ of it. Applying (1), we can rearrange (3) as

$$Y(t) = I(t) + (1 + \tau_C) C(t) + \left[ \alpha \tau_K + (1 - \alpha) \tau_L - \frac{(1 - \alpha) su(t)}{1 - u(t) - l(t)} \right] Y(t).$$
Here, the after-tax physical capital income is \((1 - \tau_K) \alpha Y (t)\). The after-tax human capital income in goods production is \((1 - \tau_L) (1 - \alpha) Y (t)\). Notice that the human capital in human capital production does not pay labor income taxes. An interpretation is that the subsidy is in net terms after labor income taxes are paid. Hence, each unit of human capital earns the value of the marginal product in human capital production, plus the subsidy, \(sw (t) A (t)\).

Government revenue after subsidies is equal to the government consumption, denoted by \(G (t) > 0\) for all \(t\), which is given exogenously. That is,

\[
(7) \quad \tau_C C (t) + \left[ \alpha \tau_K + (1 - \alpha) \tau_L - \frac{(1 - \alpha) su (t)}{1 - u (t) - l (t)} \right] Y (t) = G (t).
\]

Finally, the representative consumer has preferences over consumption and leisure, with a constant-relative-risk-aversion parameter, \(\sigma\), where \(\sigma > 0\) and \(\sigma \neq 1\). As \(\sigma \to 1\), the preferences approach to the log utility. An equilibrium of this economy is defined as follows:

**Definition 1**  
An equilibrium of the economy is a path of 
\(\{C (t), h (t), \bar{h} (t), K (t), u (t), l (t), Y (t)\}_{t=0}^{\infty}\) such that

(a) the representative consumer solves

\[
(8) \quad \max_{\{C (t), h (t), h (t)\}_{t=0}^{\infty}} \int_{0}^{\infty} e^{-\rho t} \left( \frac{(1 - \sigma)}{1 - \rho} \right) C (t) \frac{dt}{L (t)} dt,
\]

for \(\sigma > 0\) and \(\sigma \neq 1\), satisfying \(C (t) > 0\), \(0 < l (t) < 1\) and \(0 < u (t) < 1\), subject to

\[
(9) \quad \dot{K} (t) = \left[ 1 - \alpha \tau_K - (1 - \alpha) \tau_L + \frac{(1 - \alpha) su (t)}{1 - u (t) - l (t)} \right] Y (t) - (1 + \tau_C) C (t) - \delta_K K (t),
\]

and

\[
(10) \quad \dot{h} (t) = B (t) [u (t) h (t)]^\phi \bar{h} (t)^\theta - (\delta_h + g_L) h (t),
\]

for all \(t > 0\), with \(Y (t) = K (t)^\alpha [A (t) [1 - u (t)] h (t) L (t)]^{1-\alpha}\), \(\dot{A} (t) = g_A A (t)\), \(\dot{B} (t) = g_B B (t)\), and \(\dot{L} (t) = g_L L (t)\), given \(\{s, \tau_C, \tau_K, \tau_L\}, \{K (0), h (0), A (0), L (0)\}\) and \(\{\bar{h} (t)\}\) for \(t \geq 0\), for \(0 \leq \delta_K < 1\), \(\phi > 0\), \(\theta > 0\), \(\phi + \theta < 1\), \(0 \leq \delta_h < 1\), \(g_L \geq 0\), \(0 < \alpha < 1\), \(g_A \geq 0\), \(g_B \geq 0\), \(K (0) > 0\), \(h (0) > 0\), \(A (0) > 0\), \(L (0) > 0\), and \(\bar{h} (t) > 0\) for all \(t > 0\),

(b) the fiscal policy balances the budget, i.e., \(7\) holds, given \(\{G (t)\}\), and

(c) a condition \(\bar{h} (t) = h (t)\) holds.
2.2. Equilibrium Conditions on the Balanced Growth Path

We focus on a balanced growth path of an equilibrium in which (i) all variables grow at constant rates (or stay at constant levels), (ii) $u(t)$ and $l(t)$ are constant, and (iii) $G(t)/Y(t)$ is constant at $\gamma$. The second restriction is reasonable: If $u(t)$ has a strictly positive growth rate, then it will eventually violate $u(t) < 1$. If $u(t)$ has a strictly negative growth rate, then it will converge to 0, which is not consistent with the data. Similar arguments apply for $l(t)$.

The equilibrium on the balanced growth path is solved for in Appendix. Here, we present and economically interpret the conditions in the equilibrium on the balanced growth path and manipulate them as a convenient form for the calibration. Time indicators are omitted from here unless required. Throughout this paper, $g_X$ denotes a growth rate of a variable $X$ on the balanced growth path. The growth rates are related as:

$$(11) \quad g_Y = g_C = g_K = g_A + g_L + g_h,$$

$$(12) \quad g_B = (1 - \phi - \theta) g_h.$$

The two laws of motions of physical capital and human capital imply

$$(13) \quad g_K + \delta_K = i / (K/Y),$$

$$(14) \quad g_C - g_A = B_0 h_0^{\phi+\theta-1} u^\phi - \delta_h,$$

where $i \equiv (Y - C) / Y$ is the investment-output ratio. We have $B_0 \equiv B(0)$ and $h_0 \equiv h(0)$ to fix a specific period for convenience. At this period 0, $A(0)$ is normalized to 1 without loss of generality. Since $K$ and $Y$ grow at the same rate from (11), $(K/Y)$ is constant in (13). The government budget constraint, (7), implies

$$(15) \quad \left[ 1 - \alpha \tau_K - (1 - \alpha) \tau_L + \frac{(1 - \alpha) su}{1 - u - l} - i \right] \frac{\tau_C}{1 + \tau_C} + \alpha \tau_K + (1 - \alpha) \tau_L - \frac{(1 - \alpha) su}{1 - u - l} = \gamma.$$
The first-order conditions are

\begin{align}
\rho + \sigma (g_C - g_L) &= \left[ 1 - \alpha \tau_K - (1 - \alpha) \tau_L + \frac{(1 - \alpha) su}{1 - u - l} \right] r - \delta_K \\
(16) \\
&= \left[ 1 - \alpha \tau_K - (1 - \alpha) \tau_L + \frac{(1 - \alpha) su}{1 - u - l} \right] \frac{\alpha}{K/Y} - \delta_K \\
(17) \\
&= (1 - l) \left[ 1 - \alpha \tau_K - (1 - \alpha) \tau_L - \frac{asu}{1 - u - l} \right] B_0 \phi u^{\phi - 1} h_0^{\phi + \theta - 1} + g_A - \delta_h \\
(18) \\
&= (1 - l) \left[ 1 - \alpha \tau_K - (1 - \alpha) \tau_L - \frac{asu}{1 - u - l} \right] \frac{(1 - \alpha) (K/Y)^{\alpha/(1 - \alpha)}}{V_0} + g_A - \delta_h \\
(19) \\
&= \frac{1 - u - l}{l} = \left( \frac{1 - \alpha}{\kappa} \right) \left[ \frac{1 - \alpha \tau_K - (1 - \alpha) \tau_L - \frac{asu}{1 - u - l}}{1 - \alpha \tau_K - (1 - \alpha) \tau_L + \frac{(1 - \alpha) su}{1 - u - l} - i} \right] \\
(20)
\end{align}

Economic interpretations of (16) through (20) follow. Equation (16) is an Euler equation, which relates per-capita consumption growth to the after-tax-and-subsidy interest rate after depreciation. The term in squared brackets reflects the distortions caused by taxes and subsidies. Notice that this term already appeared in (9). Equation (17) reflects the interest rate defined in (4) on the balanced growth path.

The right-hand side of (18) is the after-tax-and-subsidy marginal product of human capital in human capital production, after depreciation ($\delta_h$) and exogenous productivity growth ($g_A$). To directly obtain this marginal product without solving for the equilibrium, differentiate $B(t) \left[ u(t) h(t) \right]^\phi \tilde{h}(t)^\theta$ with respect to $u(t) h(t)$ and impose $\tilde{h}(t) = h(t)$. The term in squared brackets reflects the distortions caused by taxes and subsidies. If $s = 0$, it collapses to one.

The right-hand side of (19) is the after-tax-and-subsidy marginal product of human capital in goods production, after depreciation ($\delta_h$) and exogenous productivity growth ($g_A$). We define $V(t)$ to be a unit value of human capital in units of physical goods, where
$V_0 \equiv V(0)$. Hence, (18) and (19) imply

$$(\text{VMP of physical capital}) = (\text{VMP of human capital in human capital production})$$

$$= (\text{VMP of human capital in goods production})$$

where VMP is the value of marginal product, after depreciations and exogenous productivity growth are controlled for.

Finally, (20) equates the marginal utilities when human capital is used for leisure and for goods production. See Appendix for the full derivation. To see the implication of (20), notice that there are $h(t)L(t)$ units of human capital in this economy. First, suppose that an additional unit of human capital is used for leisure. This implies that an additional fraction $1/(h(t)L(t))$ of human capital stock is used for leisure. Hence, the marginal benefit from the additional leisure is $[\kappa l(t)^{\kappa(1-\sigma)-1} (C(t)/L(t))^{1-\sigma} L(t)]/(h(t)L(t))$. Second, suppose that an additional unit of human capital is used for goods production. As discussed earlier, this is equivalent to additional $A(t)$ effective units. Their marginal products after taxes and subsidies are $A(t)$ multiplied by

$$\left[1 - \alpha \tau_K - (1 - \alpha) \tau_L - \frac{\alpha su}{1 - u - l}\right] (1 - \alpha) \left(\frac{K(t)}{Y(t)}\right)^{\alpha/(1-\alpha)}$$

which is also included in the right-hand side of (19). In addition, the marginal benefit from consumption is $l(t)^{\kappa(1-\sigma)} C(t)^{-\sigma} L(t)^{\sigma}/(1 + \tau_C)$ after consumption taxes are considered. Hence, equating the marginal utilities from these two uses of human capital, leisure and goods production, yields

$$\kappa l(t)^{\kappa(1-\sigma)-1} (C(t)/L(t))^{1-\sigma} L(t)$$

$$= A(t) \left[1 - \alpha \tau_K - (1 - \alpha) \tau_L - \frac{\alpha su}{1 - u - l}\right] (1 - \alpha) \left(\frac{K(t)}{Y(t)}\right)^{\alpha/(1-\alpha)} \frac{l(t)^{\kappa(1-\sigma)} C(t)^{-\sigma} L(t)^{\sigma}}{1 + \tau_C}.$$

As described in the Appendix, this implies (20). In (20), the left-hand side is the marginal rate of substitution between the human capital devoted to goods production and to leisure. This marginal rate of substitution is determined by the share of human capital income, $1 - \alpha$, as well as the preference parameter for leisure, $\kappa$. While $\tau_C$ is not
immediately visible in (20), it does affect the decision on leisure. That is, $\tau_C$ appears in (15), determining $u$, $l$, and $i$, along with other equations including (20).

The economy is characterized by ten equations (11) through (20), all in terms of constants.

### 2.3. Matching Data to the Model

We use U.S. data, over the span of 1960-2008 whenever possible. Many observations in National Income and Product Accounts (NIPA) start in 1960. Year 2008 was chosen as an endpoint for two reasons. First, some data from the National Center for Educational Statistics is only available through 2008. Second, both 1960 and 2008 are years in which the economy was in a recession which ended the next year, according to the NBER business cycle data.

Goods production, $Y(t)$, does not coincide with GDP. This is because GDP includes aspects of the human capital production, such as education or R&D. The objective of these activities is to accumulate knowledge rather than to produce physical goods. Hence, we first exclude them from GDP to measure $Y(t)$. Through the same method as Choi (2011), approximately 93 percent of GDP is devoted to goods production.

We follow Choi (2011) for the calibrations of $\alpha = 0.35$, $g_Y = g_K = g_C = 3.2\%$, $g_L = 1.1\%$, $i = 0.21$, $\delta_K (K/Y) = 0.12$, $\delta_h = 3.5\%$, $\tau_K = 0.43$, $\tau_L = 0.25$, and $\tau_C = 0.06$. See Part (A) of Table 1 for the calibrated values.

We assume $g_h = (1/2) (g_Y - g_L)$ as a benchmark. This implies that human capital accumulation is responsible for a half of per-capita GDP growth. This is between the two extremes $g_h = (1/3) (g_Y - g_L)$ and $g_h = g_Y - g_L$. The first extreme, in which human capital accumulation is responsible for a third of per-capita GDP growth, is based on the empirical studies which do not introduce human capital externalities. For example, see Rangazas (2005). The second extreme, in which human capital accumulation is a sole contributor to per-capita GDP growth, is assumed in numerous studies in labor literature. For example, see Heckman, et al. (2008). Later, in Table 3, we discuss how the results are sensitive to the choice of $g_h$. 

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In addition, we use $g_B = \phi g_A/2$ as benchmark. This is the mid-point of two extreme values for $g_B$. One extreme is $g_B = 0$, implying that the productivity in human capital production is constant. The other extreme is $g_B = \phi g_A$, in which the human-capital-augmented productivities in human capital production and goods production (i.e., $B(t)^{1/\phi}$ and $A(t)$) grow at the same rate. However, we also report the calibration results based on the two extreme values as robustness later checks in Table 3.

Finally, we consider leisure. The estimate by Hendricks (1999) is $l = 0.56$. As discussed in Choi (2011), the data for the human capital income in goods production, the human capital income in human capital production, and the subsidies on human capital production, provide $u = 0.13$ and $s = 0.15$. The calibrations so far are reported in Part (A) of Table 1.

**Table 1:** Calibration Results

(A) Values used to Solve the Equation System (11) through (20).

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>K share in $Y$ production</td>
<td>0.35</td>
</tr>
<tr>
<td>$\delta_h$</td>
<td>Depreciation rate of $H$</td>
<td>3.5%</td>
</tr>
<tr>
<td>$\delta_K (K/Y)$</td>
<td>Consumption of fixed capital over $Y$</td>
<td>0.12</td>
</tr>
<tr>
<td>$g_A$</td>
<td>Productivity growth in $Y$ production</td>
<td>1.1%</td>
</tr>
<tr>
<td>$g_h$</td>
<td>Growth of $H$ per worker</td>
<td>1.1%</td>
</tr>
<tr>
<td>$g_L$</td>
<td>Population growth</td>
<td>1.1%</td>
</tr>
<tr>
<td>$g_Y$</td>
<td>Growth of $Y$</td>
<td>3.2%</td>
</tr>
<tr>
<td>$i$</td>
<td>Investment rate</td>
<td>0.21</td>
</tr>
<tr>
<td>$l$</td>
<td>Fraction of $H$ devoted to leisure</td>
<td>0.56</td>
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<tr>
<td>$u$</td>
<td>Fraction of $H$ devoted to $H$ production</td>
<td>0.13</td>
</tr>
<tr>
<td>$\tau_K$</td>
<td>Tax rate on $K$ income in $Y$ production</td>
<td>0.43</td>
</tr>
<tr>
<td>$\tau_L$</td>
<td>Tax rate on $H$ income in $Y$ production</td>
<td>0.25</td>
</tr>
<tr>
<td>$\tau_C$</td>
<td>Tax rate on consumption</td>
<td>0.04</td>
</tr>
<tr>
<td>$s$</td>
<td>Subsidy rate to $H$ production</td>
<td>0.15</td>
</tr>
</tbody>
</table>
(B) Values Calibrated from the Solution to Equation System (11) through (20)

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_0 h_0^{\phi+\theta-1}$</td>
<td>Productivity in H production</td>
<td>0.1</td>
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<tr>
<td>$\delta_K$</td>
<td>K depreciation rate</td>
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<tr>
<td>$g_B$</td>
<td>Productivity growth in H production</td>
<td>0.2%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Power coefficient for direct input</td>
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</tr>
<tr>
<td>$\theta$</td>
<td>Power coefficient for externalities</td>
<td>0.59</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$G/Y$ ratio</td>
<td>0.29</td>
</tr>
<tr>
<td>$K/Y$</td>
<td>$K/Y$ ratio</td>
<td>2.85</td>
</tr>
<tr>
<td>$r$</td>
<td>Before-tax-and-subsidy interest rate</td>
<td>12.2%</td>
</tr>
<tr>
<td>$r^*$</td>
<td>After-tax-and-subsidy interest rate</td>
<td>8.9%</td>
</tr>
<tr>
<td>$r^* - \delta_K$</td>
<td>Net after-tax-and-subsidy interest rate</td>
<td>4.5%</td>
</tr>
<tr>
<td>$V_0$</td>
<td>Unit value of H when $A = 1$</td>
<td>4.83</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Power coefficient for leisure</td>
<td>1.52</td>
</tr>
</tbody>
</table>

2.4. Calibration of the Main Equations

Equation (12) becomes

$$g_B = 0.011 (1 - \phi - \theta),$$

where $g_B = \phi g_A/2$ by assumption made in the last subsection. Equations (13) and (14) become

$$0.033 + \delta_K = \frac{0.21}{K/Y} \quad \text{where } \delta_K (K/Y) = 0.12 \text{ from data},$$

$$0.033 - 0.011 = B_0 h_0^{\phi+\theta-1}0.13^\phi - 0.035,$$

Equation (15) directly provides $\gamma = 0.29$. Equations (16), (17), (18) and (19) become

$$\rho + \sigma (0.033 - 0.011) = 0.728r - \delta_K$$

$$= 0.728 \times \frac{0.35}{K/Y} - \delta_K$$

$$= 0.562\phi B_0 h_0^{\phi+\theta-1}0.28^\phi - 0.035 + 0.011$$

$$= 0.295 \times \frac{(1 - 0.35) (K/Y)^{0.35/(1-0.35)}}{V_0} - 0.035 + 0.011.$$
Finally, (20) directly provides $\kappa = 1.52$. This is close to the calibrated value for leisure parameter in a related model without human capital externalities, as reported in Gómez (2003).\textsuperscript{8}

The solutions to unknown constants in this equation system are summarized in Part (B) of Table 1. The parameter of most interest is $\theta$ since it measures learning externalities. Table 1 reports $\theta = 0.59$. This implies that learning externalities play an important role in human capital production. Hence, there is a possibility that the status quo under-produces human capital. In the next section, we study whether the welfare improves with a certain fiscal policy of $s$, $\tau_K$, $\tau_L$, and $\tau_C$, alternative to the status quo.

3. Optimal Fiscal Policy

3.1. First-Best Solution

We now consider the social planner’s problem to maximize the consumer’s discounted welfare, internalizing learning externalities. Then, we compare the planner’s solution to the status quo which we calibrated in Section 2. We further study whether fiscal policy (i.e., subsidies and taxes) can attain the planner’s solution in a competitive equilibrium.

Formally, the social planner’s problem is to solve

$$\max_{\{C(t),I(t),u(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{(l(t)\kappa C(t)/L(t))^{1-\sigma}}{1-\sigma} L(t) \, dt,$$

subject to

$$\dot{K}(t) = Y(t) - C(t) - G(t) - \delta K(t),$$

$$\dot{h}(t) = B(t) [u(t) h(t)]^\sigma h(t)^\theta - (\delta h + g_L) h(t),$$

where $Y(t) = K(t)^{\alpha} \left[ [1 - u(t) - l(t)] h(t) A(t) L(t) \right]^{1-\alpha}$ and $G(t) = \gamma Y(t)$. Solving for this problem on the balanced growth path yields solutions identical to equations (11)\textsuperscript{8} Table 2 of Gómez (2003) documents that the calibrated value for leisure parameter ($\kappa$ in this paper, or $\eta$ in Gómez (2003)) does not substantially change according to leisure specification.
through (20), with the exception of (18) and (19) which are replaced by

\[ \rho + \sigma (g_C - g_L) = (1 - l) B_0 \phi u \phi^{-1} h_0 \phi^{\phi + \theta - 1} + \theta B_0 u \phi^{\phi + \theta - 1} + g_A - \delta_h \]

\[ = (1 - l) \frac{(1 - \alpha) (K/Y)^{\alpha/(1-\alpha)}}{V_0} + \theta B_0 u \phi^{\phi + \theta - 1} + g_A - \delta_h \]

In both (28) and (29), the term \( \theta B_0 u \phi^{\phi + \theta - 1} \) is added. This term is the marginal product of human capital arising from learning externalities. This implies that the planner considers the social returns, not private returns, in his equalization of the values of marginal products.

The updated equation system is solved with the values for exogenous parameters, i.e., \( \alpha, \delta_h, \delta_K, g_A, g_L, g_B, \phi, \gamma \) and \( \kappa \), reported in Table 1. Table 2 summarizes the planner’s solutions to choice parameters, i.e., \( u, l, \) and \( i \). Notice that from (12), \( g_h \) (on the balanced growth path) is determined exogenously. Hence, the per-capita GDP growth, \( g_Y - g_L \) is also determined exogenously by (11). This implies that the planner’s solution provides the same growth rates of per-capita GDP and per-capita consumption as in a competitive equilibrium, while the levels can be different.

**Table 2:** Status Quo, Social Planner’s Solution (First Best), and Second Best

<table>
<thead>
<tr>
<th></th>
<th>Status Quo (Table 1)</th>
<th>Planner’s Solution (First Best)</th>
<th>Second Best</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>0.13</td>
<td>0.22</td>
<td>0.20</td>
</tr>
<tr>
<td>( l )</td>
<td>0.56</td>
<td>0.49</td>
<td>0.58</td>
</tr>
<tr>
<td>( i )</td>
<td>0.21</td>
<td>0.29</td>
<td>0.37</td>
</tr>
<tr>
<td>( C(0) )</td>
<td>(0.49\times10^4)</td>
<td>(2.07\times10^4)</td>
<td>(1.24\times10^4)</td>
</tr>
<tr>
<td>( Y(0) )</td>
<td>(1.00\times10^4)</td>
<td>(3.35\times10^4)</td>
<td>(2.48\times10^4)</td>
</tr>
<tr>
<td>( wel(0) )</td>
<td>-0.10</td>
<td>-0.06</td>
<td>-0.07</td>
</tr>
</tbody>
</table>

Note: \( wel(0) \) is the welfare level at period 0.

The planner’s solution can be characterized as working "smarter" but not necessarily "harder." That is, more time in human capital production is offset by less leisure time, leaving time devoted to good production relatively unchanged (0.29 instead of 0.31). In addition, the planner will invest a higher fraction of output to accumulate physical capital. This provides levels of output and consumption significantly higher than the
status quo. These results suggest some interesting conclusions. First, even with the current subsidies (which appear to be large as discussed in Section 1), the resource devoted to the production of human capital appears to be less than optimal. Second, keeping with Prescott (2004) (who focuses on income taxes), distortions from public policy appear to be causing consumers to take more leisure in the status quo.

The next question is whether this planner’s solution can be obtained in a competitive equilibrium through fiscal policy. We plug the planner’s solutions into the equations describing the competitive equilibrium, i.e., (11) through (20). Then, we solve for a combination of $\tau_K$, $\tau_L$, $\tau_C$ and $s$. This solution would become the optimal fiscal policy.

However, such a fiscal policy does not exist. To see this, notice that in the equation system (11) through (20), equations (15), (16), (17), (18), (19) and (20) contain any of $\tau_K$, $\tau_L$, $\tau_C$ and $s$. However, (16) merely determines $r$ in this competitive equilibrium after all other unknowns are solved for. Similarly, (19) determines $V_0$. This leaves four equations, (15), (17), (18) and (20), as well as four unknowns, $\tau_K$, $\tau_L$, $\tau_C$ and $s$. However, $\tau_K$ and $\tau_L$ are always seen together as $\alpha \tau_K + (1 - \alpha) \tau_L$ in all four equations. Hence, effectively, the equation system has four equations and three unknowns. It turns out that given the planner’s solutions, the equation system is over-identified.

### 3.2. Second-Best Solution

Since the first-best solution is not attainable in a competitive equilibrium, we now seek the welfare maximizing alternative. The last column of Table 2 reports the second-best solution. The welfare level is lower than the first-best solution, but higher than the status quo.

In order to understand the second-best solution more clearly, Figure 1 shows the welfare level that results from any possible combination of income tax and subsidy rates. On x-axis, we have income tax rate $(\alpha \tau_K + (1 - \alpha) \tau_L)$.$^9$ On y-axis, we have the subsidy rate $(s)$. Given these two, we solve for (i) $u$ and $l$ in representative consumer’s problem, (8), as well as (ii) the consumption tax rate $(\tau_C)$ satisfying the government’s budget

---

$^9$Notice that $\tau_K$ and $\tau_L$ are not separately identified as discussed in the previous section. Hence, we study their weighted average, $\alpha \tau_K + (1 - \alpha) \tau_L$, as income tax rate.
constraint, (15), both on the balanced growth path. In other words, a combination of an x-axis value and an y-axis value becomes a unique fiscal policy, which provides the solution for $u$ and $l$ in a competitive equilibrium. This provides the associated welfare level measured by the discounted utility in (8), which is illustrated on z-axis in Figure 1.

**Figure 1**: Fiscal Policy and the Welfare Level in a Competitive Equilibrium

Using Figure 1, we break down the analysis to study discrete policy changes. Looking at any single plane along the y-axis (which holds the value on the x-axis fixed), we are holding $s$ fixed for a variety of income tax rates. Notice that the consumption tax rate adjusts to insure that the government budget constraint is still met. Doing that, we see the welfare level decreases as we increase the income tax rate. This result is relatively familiar in the literature. See, for example, Gómez (2003) who finds positive welfare gains by replacing income tax with consumption tax across a variety of model specifications.

A relatively new finding is that when we fix the income tax rate and we increase $s$, the welfare level increases at first, and then decreases after some point. This is because there are two offsetting effects regarding the level of $s$. On one hand, as $s$ increases, the
consumer spends more time in human capital production due to subsidies. Since the consumer benefits from learning externalities, a higher production of human capital will increase the welfare level eventually. On the other hand, an increased level of subsidy discourages goods production. If goods production is discouraged extensively, it can eventually discourage human capital production which is necessary to produce goods. In addition, subsidies need to be financed by consumption taxes which decrease consumption and eventually goods production. Combining these two effects, when \( s \) is relatively low, an increase in \( s \) can increase the welfare level since the first effect is larger. When \( s \) is relatively high, the opposite is true since the second effect is larger.

The welfare level is maximized when the income tax rate is zero and \( s \) is equal to about 0.5. The status quo is also indicated in Figure 1. According to the figure, the welfare level can increase from the status quo as (i) the income tax rate decreases, or (ii) the subsidy rate increases, which is financed by an increase in the consumption tax rate.

**Figure 2:** Fiscal Policy and \( u \) in a Competitive Equilibrium
What factors define the shape of the welfare function? Figure 2 illustrates how $u$ is affected by the fiscal policy. Typically, given the subsidy rate ($s$), an increase in the income tax rate increases $u$ at first, and then decreases it after some point. This is because there are two effects. First, a higher income tax rate will decrease the time devoted to goods production (i.e., $1 - u - l$), which might indirectly increase $u$ or $l$. Second, for the choice between $u$ and $l$, $u$ is initially favored. However, since the payoff of a higher stock of human capital is taxed, $u$ eventually drops off as the income tax continues to rise.

For any given income tax rate, an increase in $s$ also increases $u$ at first, but then decreases it after some point. While a higher $s$ encourages $u$, it discourages goods production. As this discouragement strengthens, it can also discourage $u$. In addition, since leisure is not taxed while $u$ is indirectly taxed with consumption taxes, a high consumption tax may encourage leisure.

When the subsidy rates are low, the effect of income taxes on time devoted to human capital is negligible. As subsidy rates rise, however, income taxes begin to have a more pronounced effect on time devoted to human capital production. Still, high tax rates diminish the effects of subsidies. The maximum time devoted to human capital is observed in the case where income taxes are zero and subsidies are at about 50%, exactly the same point as the welfare-maximizing second-best solution.

### 3.3. Sensitivity Checks

The parameter of most interest is $\theta$ since it measures the size of the learning externality. Recall that Table 1 reports $\theta = 0.59$ as the benchmark calibration result. In this subsection, we conduct a sensitivity check of $\theta$ for alternative assumptions on parameter values. We also consider how the first-best and second-best solutions are affected.

In Subsection 2.3, we assumed $g_h = (1/2) (g_Y - g_L)$. Here, we consider two other assumptions: $g_h = g_Y - g_L$ and $g_h = (1/3) (g_Y - g_L)$. The results are summarized in Part (A) of Table 3. When $g_h = g_Y - g_L$ (i.e., the model is purely endogenous), the calibrated value for $\theta$ increases from 0.59 (benchmark) to 0.73. At the same time, the planner’s solution for $u$ increases from 0.22 (benchmark) to 0.34. This is because human capital
accumulation becomes more important. On the other hand, when $g_h = (1/3)(g_Y - g_L)$ (i.e., the model becomes more conservative), the calibrated value for $\theta$ falls from 0.59 (benchmark) to 0.44. At the same time, the first-best solution for $u$ decreases from 0.22 (benchmark) to 0.17. The results indicate that $\theta$ remains large regardless of the assumption about exogenous productivity growth, so even in the most conservative case a positive externality remains. In addition, the first-best solution for $u$ is between 0.17 and 0.34, which is above the status quo of 0.13 in all cases.

We also conduct the same sensitivity check for the second-best solution. The second-best solutions for $u$ and $l$ are not affected by alternative assumptions on exogenous growth. This is because the second-best solution is a competitive equilibrium for an individual consumer. That is, an individual consumer takes the learning externalities as exogenous, and hence, whether growth is due to learning externalities (reflected in $g_A$) does not matter in her decision of $u$ and $l$. If $g_A$ changes, then $\theta$ adjusts so that the economy provides the given growth rate $(g_Y - g_L = 2.2\%)$, but this does not matter in the individual decision.\textsuperscript{10}

Part (B) of Table 3 considers other parameters. As $g_B$ moves from zero (at the minimum) to 0.4 percent (at the maximum), the calibrated value of $\theta$ moves from 0.72 to 0.44, but it stays above zero. If human capital depreciates only due to retirement (and there is no individual depreciation), then $\delta_h = 0.022$ and $\theta = 0.56$. On the other hand, if individual depreciation is the only source of human capital depreciation (and there is no retirement), then $\delta_h = 0.013$ and $\theta = 0.54$. Hence, calibrated values for $\theta$ are substantially greater than zero even with alternative values of $\delta_h$. At the same time, the first-best solution for $u$ varies between 0.17 and 0.32, which is all above the status quo, 0.13.

The second-best solutions for $u$ and $l$ are not affected. Regarding the change in $g_B$, whether human capital production is contributed by learning externalities or exogenous productivity growth ($g_B$) does not matter for an individual consumer. The value of $\delta_h$.

\textsuperscript{10}Technically, the sensitivity checks for two cases, $g_h = g_Y - g_L$ and $g_B = 0$, are obtained as $g_h$ approaches to $g_Y - g_L$ and $g_B$ approaches to zero, respectively. This is because the welfare level approaches to infinity as $g_h \rightarrow g_Y - g_L$ or $g_B \rightarrow 0$, making it difficult to solve the welfare-maximizing problem numerically.
does not matter, either. This is because the decision on $u$ and $l$ is an allocation of human capital, while a change in $\delta_h$ affects human capital in the same manner no matter in what sector it is used: goods production, human capital production, or leisure.

Table 3: Sensitivity Checks
(Benchmark Calibration: $\theta = 0.59$, 1B $u = 0.22$, 1B $l = 0.49$)

(A) Alternative Assumptions on Exogenous Growth

<table>
<thead>
<tr>
<th>$g_h = g_Y - g_L$</th>
<th>Remark</th>
<th>Calibrated value</th>
<th>1B $u$</th>
<th>1B $l$</th>
<th>2B $u$</th>
<th>2B $l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_h = (1/3)(g_Y - g_L)$</td>
<td>Purely endogenous</td>
<td>$\theta = 0.73$</td>
<td>0.34</td>
<td>0.41</td>
<td>0.20</td>
<td>0.58</td>
</tr>
<tr>
<td></td>
<td>Conservative</td>
<td>$\theta = 0.44$</td>
<td>0.17</td>
<td>0.52</td>
<td>0.20</td>
<td>0.58</td>
</tr>
</tbody>
</table>

(B) Alternative Assumptions on Other Parameters

<table>
<thead>
<tr>
<th>$g_B = 0$</th>
<th>Remark</th>
<th>Calibrated value</th>
<th>1B $u$</th>
<th>1B $l$</th>
<th>2B $u$</th>
<th>2B $l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum $g_B$</td>
<td>$\theta = 0.72$</td>
<td>0.32</td>
<td>0.43</td>
<td>0.20</td>
<td>0.58</td>
<td></td>
</tr>
<tr>
<td>Maximum $g_B$</td>
<td>$\theta = 0.44$</td>
<td>0.17</td>
<td>0.52</td>
<td>0.20</td>
<td>0.58</td>
<td></td>
</tr>
<tr>
<td>$\delta_h = 2.2%$</td>
<td>Retirement only</td>
<td>$\theta = 0.56$</td>
<td>0.20</td>
<td>0.50</td>
<td>0.20</td>
<td>0.58</td>
</tr>
<tr>
<td>Individual depreciation only</td>
<td>$\theta = 0.54$</td>
<td>0.18</td>
<td>0.51</td>
<td>0.20</td>
<td>0.58</td>
<td></td>
</tr>
</tbody>
</table>

Note: 1B=First-Best, 2B=Second-Best

4. Concluding Remarks

This paper estimates learning externalities by calibrating a two-sector growth model using U.S. data. The results suggest that significant learning externalities exist. The time currently devoted to human capital production is lower than optimal. While the planner’s solution is unachievable in a competitive equilibrium, our analysis on the second-best solution suggests that an increased rate of subsidy on human capital production is welfare-improving. While there seems to be a role for income taxes in encouraging the growth of human capital for any given subsidy rate, the effect of income taxes seems to diminish the effects of subsidies.

The calibration in this paper estimates learning externalities only, but static externalities in goods production can be also considered. Since human capital becomes a more important contributor to economic growth, the planner’s choice on the time devoted to
human capital production is likely to increase. A higher subsidy rate will increase the welfare even more substantially.

Future work in this field could put a greater focus on the role of government. Most obvious would be endogenizing the role of government size, treated as fixed in this model with the exception of subsidies. Also worth considering is a greater focus on time consistency. With the assumption of a balanced growth path, much of the focus was placed on the state of the economy at time zero, knowing that the growth rates would be constant beyond there whatever the starting point was. However, it is necessary to consider the time consistency of the government’s objectives.
Appendix

This Appendix provides the steps to solve the model described in Subsection 2.1, in order to obtain (11) through (20) on the balanced growth path. A more detailed appendix can be found at our websites.

Since \( A(t) = e^{g_A t} A(0) \) and \( B(t) = e^{g_B t} B(0) \), we have
\[
B(t) = e^{(g_B - g_A) t} \frac{B(0)}{A(0)} A(t).
\]

Rewrite (2) as
\[
\dot{h} = e^{(g_B - g_A) t} \frac{B(0)}{A(0)} A [uh]^{\phi} \tilde{h}^\theta - (\delta_h + g_L) h.
\]

To write the problem in an intensive form, define
\[
c \equiv \frac{C}{A^{\frac{1}{1-\phi-\theta}} + 1} L,
\]
\[
k \equiv \frac{K}{A^{\frac{1}{1-\phi-\theta}} + 1} L,
\]
\[
\tilde{h} \equiv \frac{h}{A^{\frac{1}{1-\phi-\theta}}},
\]

Then,
\[
\dot{k} = \left[ d + \frac{(1 - \alpha) su}{1 - u - l} \right] k^\alpha (1 - u - l)^{1-\alpha} \tilde{h}^{1-\alpha} - (1 + \tau_C) c - ek,
\]

where \( e \equiv \delta_K + \left( \frac{g_B}{g_A} \frac{1}{1-\phi-\theta} + 1 \right) g_A + g_L \). In addition,
\[
\dot{\tilde{h}} = B u^{\phi} \tilde{h}^{\phi} \tilde{h}^{-\theta} - f \tilde{h},
\]

where \( f \equiv \delta_h + g_L + \frac{g_B}{1-\phi-\theta} \). The objective becomes
\[
\max_{\{C,L,u\}} \int_0^\infty e^{-\eta t} \left( \frac{(k^c)^{1-\sigma}}{1-\sigma} \right) dt
\]

where \( \eta \equiv \rho - g_A \left( \frac{g_B}{g_A} \frac{1}{1-\phi-\theta} + 1 \right) (1 - \sigma) - g_L \). Here, \( \eta > 0 \) must be satisfied with calibrated values.

The Hamiltonian function becomes
\[
H = \frac{(k^c)^{1-\sigma}}{1-\sigma} + \lambda \left[ d + \frac{(1 - \alpha) su}{1 - u - l} \right] k^\alpha (1 - u - l)^{1-\alpha} \tilde{h}^{1-\alpha} - (1 + \tau_C) c - ek + \mu \left[ B u^{\phi} \tilde{h}^{\phi} \tilde{h}^{-\theta} - f \tilde{h} \right]
\]
The first-order conditions are \( H_c = 0, H_l = 0, H_u = 0, H_k = \eta \lambda - \dot{\lambda}, \) and \( H_h = \eta \mu - \dot{\mu}. \) Rearranging these first-order conditions, as well as two constraints, we have

\[
\begin{align*}
(30) \quad 0 &= l^\kappa (l^\kappa c)^{-\sigma} - (1 + \tau_C) \lambda, \\
(31) \quad 0 &= \kappa l^{\kappa-1} c (l^\kappa c)^{-\sigma} - (1 - \alpha) \left( d - \frac{\alpha s u}{1 - u - l} \right) (1 - u - l)^{-\alpha} k^\alpha \bar{h}^{1-\alpha} \lambda, \\
(32) \quad 0 &= -\lambda \left[ d - s - \frac{\alpha s u}{1 - u - l} \right] (1 - \alpha) k^\alpha (1 - u - l)^{-\alpha} \bar{h}^{1-\alpha} + \mu \bar{B} \phi u^{\phi-1} \bar{h}^{\phi-\theta}, \\
(33) \quad \eta \lambda - \dot{\lambda} &= \lambda \left( d + \frac{(1 - \alpha) s u}{1 - u - l} \right) \alpha k^{\alpha-1} (1 - u - l)^{1-\alpha} \bar{h}^{1-\alpha} - e, \\
(34) \quad \eta \mu - \dot{\mu} &= \lambda \left( d + \frac{(1 - \alpha) s u}{1 - u - l} \right) k^\alpha (1 - u - l)^{1-\alpha} (1 - \alpha) \bar{h}^{-\alpha} + \mu \left( \bar{B} u^{\phi} \bar{h}^{\phi-1} \bar{h}^{-\theta} - f \right), \\
(35) \quad \dot{k} &= \left[ d + \frac{(1 - \alpha) s u}{1 - u - l} \right] k^\alpha (1 - u - l)^{1-\alpha} \bar{h}^{1-\alpha} - (1 + \tau_C) c - e k, \\
(36) \quad \dot{h} &= \bar{B} u^{\phi} \bar{h}^{\phi-\theta} - f \bar{h}.
\end{align*}
\]

Consider a balanced growth path in which all variables grow at constant rates (or stay at constant levels). We first show \( g_c = g_k = g_h = g_\lambda = g_\mu = 0. \) Equation (30) implies

\[
(37) \quad \kappa (1 - \sigma) g_l - \sigma g_c = g_\lambda.
\]

Equation (36) implies

\[
\dot{h}/\bar{h} = \bar{B} u^{\phi} \bar{h}^{\phi+\theta-1} - f.
\]

Here, we further assume that \( u \) and \( l \) are constant. If \( u \) has a strictly positive growth rate, then it will eventually violate \( u < 1. \) If \( u \) has a negative growth rate, then it will converge to 0, which is not consistent with the data. The same reasoning applies to the assumption that \( l \) is constant. Then, the above equation implies that we either have a constant \( \bar{h} \) or \( \phi + \theta = 1. \) Since we have already assumed that \( \phi + \theta < 1, \bar{h} \) must be constant. Hence,

\[
0 = \bar{B} u^{\phi} \bar{h}^{\phi+\theta-1} - f.
\]
Equation (33) implies
\[ \eta - \dot{\lambda}/\lambda = \left[ d + \frac{(1 - \alpha) su}{1 - u - l} \right] \alpha k^{\alpha-1} (1 - u - l)^{1-\alpha} \tilde{h}^{1-\alpha} - e. \]

Hence, \( g_k = g_{\tilde{h}} = 0 \). Equation (35) implies
\[ 0 = \left[ d + \frac{(1 - \alpha) su}{1 - u - l} \right] k^\alpha (1 - u - l)^{1-\alpha} \tilde{h}^{1-\alpha} - (1 + \tau_C) c - ek. \]

Hence, \( g_c = 0 \). Equation (32) implies that after the representative consumer constraint is imposed
\[ \mu \tilde{B} \phi u^{\phi-1} \tilde{h}^{\phi+\theta} = \lambda \left( d - s - \frac{\alpha su}{1 - u - l} \right) (1 - \alpha) k^\alpha (1 - u - l)^{-\alpha} \tilde{h}^{1-\alpha}. \]

Hence, \( g_\lambda = g_\mu = 0 \).

Eliminating \( \lambda \) and \( \mu \) in the equation system of (30) through (36), we have the following five equations:

\[ \begin{align*}
(38) & \quad \kappa \frac{c}{l} = \frac{(d - \frac{\alpha su}{1 - u - l})(1 - \alpha)(1 - u - l)^{-\alpha} k^{\alpha} \tilde{h}^{1-\alpha}}{(1 + \tau_C)}, \\
(39) & \quad \eta = \left( d + \frac{(1 - \alpha) su}{1 - u - l} \right) \left( \frac{k}{\tilde{h}} \right)^{\alpha-1} \alpha (1 - u - l)^{1-\alpha} - e, \\
(40) & \quad \eta = \tilde{B} \phi u^{\phi-1} \tilde{h}^{\phi+\theta-1} \frac{(1 - l)(d - \frac{\alpha su}{1 - u - l})}{d - s - \frac{\alpha su}{1 - u - l}} - f, \\
(41) & \quad 0 = \left[ d + \frac{(1 - \alpha) su}{1 - u - l} \right] k^\alpha (1 - u - l)^{1-\alpha} \tilde{h}^{1-\alpha} - (1 + \tau_C) c - ek, \\
(42) & \quad 0 = \tilde{B} u^{\phi} \tilde{h}^{\phi+\theta-1} - f.
\end{align*} \]

Now we convert back from the intensive form to our original notation, eliminating \( c \), \( g_c \), \( k \), \( \tilde{h} \), \( \eta \), \( d \), \( e \), \( f \), and \( \tilde{B} \). It is easy to see that \( g_Y = g_C = g_K = g_A + g_h + g_L \) and \( g_B + (\phi + \theta - 1) g_h = 0 \). These become (11) and (12).

One can rearrange (39) to have (17). Using the definition of \( r \), (16) follows. Further, rearrange (40) to have (18). In addition, defining \( V \equiv \frac{a}{x} A \), (19) follows. Equation (38) becomes
\[ \kappa = \left[ \frac{1 - \alpha \tau_K - (1 - \alpha) \tau_L - \frac{\alpha su}{1 - u - l}}{(1 - u - l) (1 + \tau_C) C} \right] \frac{(1 - \alpha) Y}{(1 - u - l)} \frac{l}{(1 + \tau_C) C}. \]
where \((a)\) is the after-tax-and-subsidy marginal product of human capital in goods production. Rearranging,

\[
\frac{1 - u - l}{l} = \frac{(1 - \alpha)}{\kappa} \left[ 1 - \alpha \tau_K - (1 - \alpha) \tau_L - \frac{su}{1 - u - l} \right] \frac{Y}{(1 + \tau_C) C}
\]

From (6), we can divide by \(Y\) and have:

\[
1 = i + \frac{(1 + \tau_C) C}{Y} + \alpha \tau_K + (1 - \alpha) \tau_L - \frac{(1 - \alpha) su}{1 - u - l}
\]

Solve for \(Y / (1 + \tau_C) C\) and (20) follows.

The remaining two equations, (41) and (42), become (13) and (14).
References


