Implicit Netput Functions

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**Implicit Netput Functions**

There is a tradition in much of the physical and social sciences that envisions nature in terms of differential equations (John 1981). In a variety of settings, economic theory and applications draw on the theory of differential equations (e.g., Luenberger 1992). The widely used envelope theorem, whether static or dynamic, is strikingly simple and yet profoundly facilitates the ability to model complex interactions among variables. However, typical envelope theorem based empirical applications assume a form for the value function and then derive behavioral responses.

In this paper, we consider standard static production theory (netputs) using envelope results from unknown restricted profit functions (McFadden 1978). We stray considerably from the conventional envelope approach applied in production economics by beginning with netput functions. We assume that behavior is modeled as first order partial differential equations *in terms of all the data at hand*. That is, the forms considered consist of netputs as functions of prices, fixed inputs, and restricted profit. We characterize these as implicit netput functions because they depend not only on prices and fixed inputs but also on restricted profit itself. When this is done, a set of alternatives to the conventional approach is apparent which we believe has value added. We then make these equations integrable and aggregable drawing heavily from the theory of continuous groups on a manifold (Lie 1888).

After placing the implicit approach in context, the pros and cons are considered with the conclusion that there is much to commend the implicit approach. One advantage is that there exists a large untapped literature from demand theory that can be transported to
production applications. Then, a family of convenient implicit netput functions (linear in functions of profit) is developed in the third section using group theory. These systems are integrable and can be made to be aggregable in the Gorman sense. An application to aggregate U.S. agriculture is presented in the penultimate section followed by the final concluding section.

**An Implicit Formulation**

A common representation since at least the 1970s of a competitive firm or sector relies on the restricted profit function \( \pi(p, z) \) as a function of prices \( p \in \mathbb{R}_{++}^n \) and fixed netputs (usually inputs) \( z \in \mathbb{R}^m \), where \( \pi(p, z) = \max_{y} \left\{ p'y : y \in T(z) \right\} \) is assumed to exist where \( T(z) \) is the convex production set. Restricted profit, \( \pi \), is assumed to be: continuous on the interior and monotonically increasing in output prices and monotonically declining in input prices, linear homogeneous, and convex in \( p \) and concave in \( z \) (McFadden 1978). We will also assume that \( \pi \) is smooth in its arguments.

For applied researchers, the key link of \( \pi \) to behavior occurs by use of the envelope theorem which is the key partial differential equation governing behavior:

\[(1) \quad y = \frac{\partial \pi(p, z)}{\partial p},\]

where \( y \) represents netputs (inputs < 0 and outputs > 0). This creates, as opposed to first order conditions of a primal problem, a commonly interpreted reduced form with the endogenous variables (netput choices) on the left hand side of (1) and the predetermined or exogenous variables on the right hand side (at least for individual firm applications). There can be no doubt that the success of the duality revolution in part hinges on this
convenience. Symmetry or mathematical integrability is built into such an approach in (1) if one starts with a known smooth form for \( \pi \). Some portions of economic integrability, (curvature, homogeneity, and monotonicity) are often ignored in empirical work (examples of exceptions are Dievert and Wales 1995; LaFrance, Pope, and Tack 2011).

The data primitives in order to estimate (1) are usually considered to be \((p, z, y)\) but more generally can be thought of as \((p, z, y, \pi)\). Indeed, it is often the case that profit is calculated by summing any receipts (revenue) and subtracting expenses without any taxonomy as to input or output categories. This is increasingly true in firm-level data sets. Thus, in many instances \( \pi \) is a primitive and could conceivably be measured more precisely than by taking the inner product of \( p \) and \( y \). In any case, in this paper we will consider \( \pi \) as data.

Hence, a partial differential equation approach based on all available data could be of the form:

\[
y = \frac{\partial \pi}{\partial p} = g(p, z, \pi).
\]

As opposed to (1), we say netputs in (2) are in implicit form. Note that if one starts with the usual notion of partial differential equations that are typically solved in mathematics, it would be very natural to include \( \pi \) on the right hand side.

Consider the translog share functions for quantities demanded and supplied. Then the netput functions can be written in the form:

\[
y = \Delta(p^{*})\left[\zeta + B \ln(p) + C \ln(z)\right] \pi,
\]

where \( \Delta(p^{*}) \) is a diagonal matrix that contains the reciprocals of netput prices.
(1/p_1, ..., 1/p_n) down the main diagonal. Thus, netputs (as opposed to shares) are in implicit form. In this case, we can construct a version of the netput own-price response holding profit fixed,

$$\frac{\partial y_i}{\partial p_i} = -\left(\frac{1}{p_i}\right) \left[ \gamma_i + \sum_{j=1}^{n} b_{ij} \ln p_j + \sum_{k=1}^{m} c_{ik} \ln z_k \right] \pi + \left(\frac{1}{p_i^2}\right) b_{ii} \pi$$

(4)

$$= \frac{y_i}{p_i} \left( \frac{b_{ii}}{s_i} - 1 \right), \ i = 1, ..., n,$$

where $s_i = p_i y_i / \pi$ is the profit share of the $i$th netput, and the ordinary own-price response,

$$\frac{\partial y_i}{\partial p_i} = -\left(\frac{1}{p_i^2}\right) \left[ \gamma_i + \sum_{j=1}^{n} b_{ij} \ln p_j + \sum_{k=1}^{m} c_{ik} \ln z_k \right] \pi + \left(\frac{1}{p_i^2}\right) b_{ii} \pi$$

(5)

$$+ \Delta (p_{ij}^{-1}) \left[ \gamma_i + \sum_{j=1}^{n} b_{ij} \ln p_j + \sum_{k=1}^{m} c_{ik} \ln z_k \right] \frac{\partial \pi}{\partial p_i}$$

$$= \frac{y_i}{p_i} \left( \frac{b_{ii}}{s_i} - 1 + s_i \right), \ i = 1, ..., n,$$

This occurs in a relatively simple and straightforward way compared to translog netputs written in terms of $p$ and $z$ and identifies the meaning of $(y_i / p_i)s_i$ in the translog model.

Therefore, one can merely say that the response to a change in own-price is a relatively simple expression in prices, fixed inputs, the levels of the netput and restricted profit.

**The Pros and Cons of the Implicit Form**

We turn next to the issue of parsimony and again use translog netputs, but only to illustrate an essential point. Equation (3) written in explicit netput form is:
where the parameters $(A, \sigma, D)$ are added to complete $\ln(\pi)$ as in the customary second-order expansion in $\ln(p)$ and $\ln(z)$. This is a fairly tedious expression that is not likely to be estimated; the share form is clearly more parsimonious. Note that every single netput equation in (6) contains all of the parameters: $\zeta$ is of dimension $n$, $B$ is of dimension $\frac{1}{2}n(n+1)$ [assumed symmetric], $C$ is $mn$, $\sigma$ is $m$, $D$ is $\frac{1}{2}m(m+1)$ [assumed symmetric], and $\tilde{A}$ is a singleton, for a total of $\frac{1}{2}(n+1)(n+2) + m(n+1) + \frac{1}{2}m(m+1)$ parameters in each netput equation.

In implicit form, each netput equation contains $m+n+1$ parameters (see (3) above). Thus, the explicit form in (6) contains $\frac{1}{2}n(n+1)+nm+\frac{1}{2}m(m+1)$ additional parameters, illustrating that an implicit form like (3) generally is more parsimonious in parameters than (6). That is, profit whether on the right hand side as in (3) or on the left hand side, creating a share equation, saves parameters when compared to explicit representations.

Having shown that implicit functional forms for netputs are common and possess desirable properties, we turn to two additional issues for implicit netputs. The first addresses the question “How does one proceed from attractive forms of netput equations in some domain in order to make them integrable?” We are guided by the modern theory of exact aggregation in consumer demand theory, adapted from the mathematics of continuous transformation groups on a manifold to obtain integrability, as pioneered by Sophus Lie (Olver 1986) and elucidated in economics by Russell (1983, 1996), Russell and Farris (1993,1998), and Jerison (1993).
Though there are conveniences to the implicit form as enumerated above, the second concern is that the implicit form in (3) may seem like it vitiates the convenience of duality – profit in (3) is clearly endogenous. It places choice variables on both the left and right hand sides of (3) because profit contains netput choices. Yet, from a modern econometric point of view, this is of little consequence so long as the appropriate instruments can be found, because prices are often already treated as endogenous in many aggregate applications. For example, in the spirit of GMM estimation, one could state a vector of moment equations of the form:

\[
\frac{1}{T} \sum_{t=1}^{T} x_t [y_t - g(p_t, z_t, \pi_t)] = \frac{1}{T} \sum_{t=1}^{T} x_t \epsilon_t, \quad t = 1, \ldots, T, \]

where \( \epsilon_t \) is a random error vector and \( x_t \) is an instrument. Therefore, one might start with the presumption of endogeneity even in (1) and search for the appropriate orthogonality or moment conditions in order to design an estimator. This seems to be increasingly the approach used in empirical work in economics. Indeed, modern applications of consumer demand often have some form of expenditure, share, or demands on the left hand side of regressions and total expenditure on the right. This necessitates a method to deal with the endogeneity in much the same way as required for implicit netputs (e.g., LaFrance 1991; Lyssiotou, Pashardes, and Stengos 1999).

Inherited Properties of Implicit Netputs
Netputs inherit properties from the restricted profit function. In particular, from the homogeneity and convexity properties of \( \pi, y \) is homogeneous of degree zero in prices and the matrix \( \frac{\partial y}{\partial \pi^T} \) is symmetric and positive semidefinite. From (2) zero degree ho-
mogeneity requires \( g_p^r, p + g_y \pi = 0 \). Thus, \( g \) is homogeneous of degree zero in \( p \) and \( \pi \). Further, \( \partial y / \partial p^r = g_p^r + g_y \pi \) must be positive semidefinite, due to convexity, and symmetry is implied by smoothness of \( \pi \) in prices.\(^4\) Note that by considering \( y \) as a vector of Hicksian consumer demands, and \( \pi \) as an expenditure or cost function, symmetry looks mathematically identical to Slutsky symmetry for compensated consumer demands.\(^5\) Therefore we have the following result:

**Proposition 1:** The homogeneity, adding up, and convexity properties of the restricted profit function in prices are expressed in terms of the implicit netput functions, \( y = g(p, z, \pi) \), respectively as: a) \( g_p^r, p + g_y \pi = 0 \); b) \( p^r g = \pi \); and c) \( g_p^r + g_y \pi \) symmetric, positive semidefinite. These properties are identical to those found for consumer demands (with the convexity of profit in netput prices replacing the concavity of consumer expenditure in goods prices), where \( y \) represents Hicksian compensated demands, \( \pi \) is total expenditure, and \( p \) represents consumer prices.

It is apparent that one can look to the theory of integrability of consumer demands rather than “reinvent the wheel” for production.

**Heterogeneity and Aggregation**

From our point of view, whether one is focused on aggregation or not, aggregation theory has led to many useful functional forms. For example, aggregation was the focus of Gorman (1981) and out of it came the AIDS (Deaton and Muellbauer 1980) which has been used successfully for both individual level and aggregate level data. For empirical pro-
duction applications which focus on aggregation, one must consider the kinds and form of heterogeneity in the netput functions: \( y = g(p, z, \pi) \). As a start, it seems reasonable to consider \( p \) homogeneous. In this case, consider Gorman’s (1981; also see Lau 1982) specification for exact aggregation adapted from consumer theory:

\[
y_i(p, z, \pi) = \sum_{k=1}^{K} \alpha_{ik}(p) f_k(p(z), z), \quad i = 1, \ldots, n.
\]

In matrix form with obvious notation:

\[
y = A(p)f(\pi, z),
\]

where \( y \) is \( n \times 1 \), \( A \) is \( n \times K \), and \( f \) is \( K \times 1 \). Restrictions on \( A \) are required for homogeneity and can be obtained once the \( f_k, k = 1, \ldots, K \), are specified. The rank of \( A \) assuming all \( f_k \) functions are linearly independent is the rank of a Gorman system. Gorman (1981) and Russell and Farris (1993, 1998) have shown that the rank of \( A \) cannot exceed three.\(^6\)

One advantage of an exactly aggregated Gorman system is that aggregate netputs are of the form:

\[
\bar{y} = A\bar{f},
\]

where the bars denote the sum or average across agents with heterogeneity in \((\pi, z)\). If \( f \) includes power functions, moments or fractional moments of profit will enter aggregate netput behavior. However, in some applications, it may be convenient to deviate from (10) and allow some or all parts of \( A \) to depend on \( z \) as well.

**Integrability with Aggregability**

Convexity requires that \( \partial y / \partial p^\tau = g_{p^\tau} + g_{\pi}y^\tau \) is a positive semidefinite matrix, while
symmetry from (9) requires that:

\[
\frac{\partial y_i}{\partial p_j} = \frac{\partial g_i}{\partial p_j} + \frac{\partial g_i}{\partial \pi} \frac{\partial \pi}{\partial p_j} \quad \text{symmetric} \quad \Rightarrow \quad \sum_{k=1}^{K} \left( \frac{\partial \alpha_{ik}}{\partial p_j} f_k + \alpha_{ik} \frac{\partial f_k}{\partial \pi} y_j \right) \quad \text{symmetric}.
\]

As noted in Proposition 1, the expression on the right hand side of the equals sign illustrates the resemblance of (11) to the Slutsky equation of consumer theory with \( \pi \) replacing income or total expenditure. The last two terms on the right hand side of the equals sign are analogous to the income effect in consumer theory when we replace income with profit. Substituting from (8) for \( y_j \) in (11) and rearranging terms then gives

\[
\sum_{k=1}^{K} \sum_{k'=1}^{K} \left( \alpha_{ik} \alpha_{ik'} - \alpha_{jk} \alpha_{ik'} \right) \frac{\partial f_k}{\partial \pi} f_{k'} = \sum_{k=1}^{K} \left( \frac{\partial \alpha_{jk}}{\partial p_i} - \frac{\partial \alpha_{ik}}{\partial p_j} \right) f_k \quad \forall \ i, j = 1, \ldots, n.
\]

The terms \( \alpha_{ik}, \forall \ i = 1, \ldots, n, \forall \ k = 1, \ldots, K, \) and \( \frac{\partial \alpha_{ik}}{\partial p_j}, \forall \ i, j = 1, \ldots, n, \forall \ k = 1, \ldots, K, \) in (12) depend on \( p \) but not on \( (\pi, z) \). On the other hand, the terms \( f_k, \forall = 1, \ldots, K, \) and \( \frac{\partial f_k}{\partial \pi}, \forall k, k' = 1, \ldots, K, \) in (12) depend on \( (\pi, z) \) but not on \( p \). This means that the available literature on integrability of consumer demand systems can be applied to implicit netputs. Indeed, nowhere is the theory of integrability as well developed as it is in the large literature on aggregation spawned by Gorman and extended by many researchers. A partial list includes: Gorman (1981), Lau (1982), Russell (1983, 1996), Muellbauer (1975, 1976), Deaton and Muellbauer (1980), Howe, Pollak, and Wales (1979), Van Daal and Merkies (1989), Lewbel (1987, 1988, 1989, 1990, 1991, 2003), LaFrance (2004), LaFrance, Beatty, and Pope (2005), and LaFrance and Pope (2009).

proved that the rank cannot exceed three in (9), but also that there exist only six possible cases under symmetry, adding up, homogeneity, and full column rank of \( A \), when the vector-valued functions \( f \) in (9) are specified in terms of nominal \( \pi \). For the sake of brevity, we omit detailed derivations. The representations given here are adapted from Van Daal and Merkies (1989), Lewbel (1990), and LaFrance and Pope (2009) to implicit input functions.\(^8\)

i. Rank 1 (Homothetic)

\[
\begin{align*}
\mathbf{y}(p, z, \pi) &= \frac{\partial \beta(p, z)}{\partial p} \pi \\
\mathbf{p}^T \frac{\partial \beta}{\partial p} &= \beta
\end{align*}
\]

Implicit Profit Function

\[
H\left(\frac{\pi}{\beta(p, z)}, z\right) = 0, \frac{\partial H(x, z)}{\partial x} > 0.
\]

ii. Rank 2 PIGL

\[
\begin{align*}
\mathbf{y}(p, z, \pi) &= \frac{\partial \beta(p, z)}{\partial p} \pi + \beta(p, z)^\kappa \frac{\partial \alpha(p, z)}{\partial p} \pi^{1-\kappa} \\
\mathbf{p}^T \frac{\partial \alpha}{\partial p} &= 0, \mathbf{p}^T \frac{\partial \beta}{\partial p} = \beta, \kappa \neq 0
\end{align*}
\]

Implicit Profit Function

\[
H\left(\frac{1}{\kappa}\left(\frac{\pi}{\beta(p, z)}\right)^\kappa - \alpha(p, z), z\right) = 0, \frac{\partial H(x, z)}{\partial x} > 0.
\]

iii. Rank 2 PIGLOG

\[
\begin{align*}
\mathbf{y} &= \frac{\partial \beta(p, z)}{\partial p} \pi + \frac{\partial \alpha(p, z)}{\partial p} \pi \ln \left(\frac{\pi}{\beta(p, z)}\right) \\
\mathbf{p}^T \frac{\partial \alpha}{\partial p} &= 0, \mathbf{p}^T \frac{\partial \beta}{\partial p} = \beta
\end{align*}
\]

Implicit Profit Function

\[
H\left(\ln \left(\frac{\pi}{\beta(p, z)}\right), \alpha(p, z), z\right) = 0, \frac{\partial H(x, z)}{\partial x} > 0.
\]

Results for the rank 3 Extended PIGL and PIGLOG cases employ on an implicit solution to a Riccati differential equation that arises in this problem (LaFrance and Pope...
2009, pp. 97-98). In cases iv. and v. below, \( \chi: \mathbb{R}_+ \times \mathbb{R}^m \rightarrow \mathbb{R} \) is an arbitrary smooth function. For brevity, and because it is seldom used, the full rank 3 trigonometric case is omitted.

iv. Rank 3 PIGL

\[
y(p, z, \pi) = \frac{\pi^{1-\kappa}}{\kappa \beta(p, z)} \left\{ \frac{\partial \alpha(p, z)}{\partial p} + \frac{\partial \beta(p, z)}{\partial p} \left( \frac{\pi^\kappa - \alpha(p, z)}{\beta(p, z)} \right) \right\} + \beta(p, z) \frac{\partial \gamma(p, z)}{\partial p} \left[ \chi(\gamma(p, z), z) + \left( \frac{\pi^\kappa - \alpha(p, z)}{\beta(p, z)} \right)^2 \right]
\]

\[p^T \partial \alpha / \partial p = \kappa \alpha, \quad p^T \partial \beta / \partial p = \kappa \beta, \quad p^T \partial \gamma / \partial p = 0, \quad \kappa \neq 0\]

Implicit Profit Function

\[
\begin{align*}
H \left( \frac{\pi^\kappa - \alpha(p, z)}{\beta(p, z)} \right) - \int_0^{y(p, z)} \left[ \chi(x, z) + w(x, z)^2 \right] dx - c(z)z & = 0 \\
\frac{\partial H(x, z)}{\partial x} & > 0
\end{align*}
\]

where

\[
w(0, z) = c(z),
\]

\[
w(\gamma(p, z), z) = \frac{\pi(p, z)^\kappa - \alpha(p, z)}{\beta(p, z)}, \quad \forall \ (p, z) \in \mathbb{R}_+^n \times \mathbb{R}^m,
\]

\[
\frac{\partial w(x, z)}{\partial x} = \chi(x, z) + w(x, z)^2, \quad \forall \ (x, z) \in \mathbb{R}_+ \times \mathbb{R}^m.
\]
\[
y(p, z, \pi) = \pi \left\{ \frac{\partial \beta(p, z)/\partial p}{\beta(p, z)} + \frac{\partial \alpha(p, z)}{\partial p} \left( \ln \left( \frac{\pi/\beta(p, z)}{\alpha(p, z)} \right) \right) \right\} \\
+ \alpha(p, z) \frac{\partial \gamma(p, z)}{\partial p} \left[ \chi(\gamma(p, z), z) + \left( \frac{\ln(\pi/\beta(p, z))}{\alpha(p, z)} \right)^2 \right] \right\} \\
p^T \partial \alpha/\partial p = 0, \quad p^T \partial \beta/\partial p = \beta, \quad p^T \partial \gamma/\partial p = 0
\]

Implicit Profit Function

\[
H \left( \frac{\ln(\pi/\beta(p, z))}{\alpha(p, z)} \right) - \gamma(p, z) \int_0^\chi(x, z + w(x, z)^2) dx - c(z), z \right) = 0
\]

\[
\frac{\partial H(x, z)}{\partial x} > 0
\]

where

\[
w(0, z) = c(z), \\
w(\gamma(p, z), z) = \frac{\ln(\pi(p, z)/\beta(p, z))}{\alpha(p, z)}, \forall (p, z) \in \mathbb{R}^n \times \mathbb{R}^m, \text{ and} \\
\frac{\partial w(x, z)}{\partial x} = \chi(x, z) + w(x, z)^2, \forall (x, z) \in \mathbb{R}_+ \times \mathbb{R}^m.
\]

The ubiquitous translog is in the rank 1 class, and moving down the cases leads to greater flexibility in \( \pi \) and rank. Adding up (assuming the system is complete) restricts one of the functions of \( \pi \) to be \( \pi \) itself. Homogeneity and adding up place restrictions on the functions \( \alpha, \beta, \gamma \). For example, in the Quadratic PIGL model, both \( \alpha \) and \( \beta \) must be homogeneous of degree \( \kappa \) in \( p \), while \( \gamma \) must be homogeneous of degree zero. If \( \pi \in \mathbb{R}_+ \), the full class of models can be used, including that \( \kappa \) can be any nonzero constant in the PIGL cases and admitting the logarithmic functional form of the PIGLOG cases.

Lau’s (1982) \textit{Fundamental Theorem of Exact Aggregation} places additional restric-
tions on the functions $\alpha, \beta, \gamma$. For example, in the rank 1 homothetic case, for this result to hold, we must have

$$\frac{\partial \beta(p, z)}{\partial p} = \sum_{l=1}^{L} g_l(z) b_l(p),$$

for some finite integer $L$. The linear independence of the $\{g_l(z)\}_{l=1}^{L}$ and the $\{b_l(p)\}_{l=1}^{L}$, along with symmetry imply that $b_l(p)=[\partial \beta_{\ell}(p)/\partial p]/\beta_{\ell}(p) \forall \ell = 1, \ldots, L$ (without loss of generality, WLOG). Direct integration to recover $\beta(p, z)$ then gives

$$\beta(p, z) = g_0(z)\prod_{l=1}^{L} \beta_{\ell}(p)^{g_{\ell}(z)}.$$

The $\{g_{\ell}(z)\}$ also are restricted by the homogeneity and adding up conditions so that $\beta(p, z)$ is homogeneous of degree one in $p$.

For Lau’s (1982) result to hold in the rank 2 PIGL and PIGLOG cases, $\beta(p, z)$ is restricted further, so that $g_{\ell}(z) = g_{\ell}$, a constant, $\forall \ell = 1, \ldots, L$, while $\alpha(p, z)$ is required to satisfy

$$\frac{\partial \alpha(p, z)}{\partial p} = \sum_{\ell'=1}^{L'} h_{\ell'}(z) a_{\ell'}(p),$$

for some finite integer $L'$. Linear independence of the $\{h_{\ell}(z)\}_{\ell'=1}^{L'}$ and the $\{a_{\ell}(p)\}_{\ell'=1}^{L'}$, plus symmetry then imply, again WLOG, that $a_{\ell}(p) = \partial \alpha_{\ell}(p)/\partial p$, $\forall \ell' = 1, \ldots, L'$. Direct integration to recover $\alpha(p, z)$ then gives

$$\alpha(p, z) = h_0(z) + \sum_{\ell'=1}^{L'} h_{\ell'}(z) a_{\ell'}(p).$$
with each of the \( \{\alpha_{\epsilon}(p)\}_{\epsilon=1}^{L'} \) homogeneous of degree zero in \( p \) by the homogeneity and adding up conditions. The conditions for Lau’s Theorem to hold in the full rank 3 cases are substantially more complicated and beyond the scope of this paper.

**Generalizations of Gorman Nominal Demand Systems**

More general representations than the Gorman system of aggregable demands exist that remain aggregable in the Gorman sense. The most notable of these is the system based on real (deflated) profit rather than nominal profit developed by Lewbel (1989, 1990). A useful approach to integrability for the Gorman/Lewbel class of aggregable demand functions is the symmetry method of Sophus Lee (1888), elucidated for consumer demand in Russell (1983, 1996), Russell and Farris (1993, 1998), LaFrance, Beatty, and Pope (2005), and LaFrance and Pope (2009). Adapting these arguments here yields netput or share equations which are linear in functions of profit and \( z \).

As in the previous section, implicitly define the smooth function \( w: \mathbb{R}_+ \times \mathbb{R}^m \rightarrow \mathbb{R} \) by the integral equation,

\[
(17) \quad w(\delta(p, z), z) = c(z) + \int_0^{\delta(p, z)} [\chi(x, z) + w(x, z)^2] dx,
\]

subject to the initial condition,

\[
(18) \quad w(0, z) \equiv c(z) \quad \forall \ z \in \mathbb{R}^m,
\]

the terminal condition,\(^9\)

\[
(19) \quad w(\delta(p, z), z) = \left[ \frac{f(\pi(p, z) + \alpha(p, z), z) - \beta(p, z)}{\gamma(p, z)} \right] \quad \forall \ (p, z) \in \mathbb{R}_+^n \times \mathbb{R}^m,
\]

and the partial differential equation,
(20) \[ \partial w(x, z)/\partial x = \chi(x, z) + w(x, z)^2, \quad \forall (x, z) \in \mathbb{R}_+ \times \mathbb{R}^m. \]

In (17)–(20), \( \alpha : \mathbb{R}_+^n \times \mathbb{R}^m \to \mathbb{R}_+ \) is 1° homogeneous in \( p \), \( \beta, \gamma, \delta : \mathbb{R}_+^n \times \mathbb{R}^m \to \mathbb{R}_+ \) are 0° homogeneous in \( p \), the non-vanishing elements of \( \{\alpha, \beta, \gamma, \delta\} \) are functionally independent in \( p \), \( f : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R} \), \( \partial f(x, z)/\partial x > 0 \) \( \forall (x, z) \in \mathbb{R}_+ \times \mathbb{R}^m \), is smooth and strictly increasing in (deflated) profit, \( \pi : \mathbb{R}_+^n \times \mathbb{R}^m \to \mathbb{R} \) is the (smooth) profit function, and \( c : \mathbb{R}^m \to \mathbb{R} \) is an arbitrary smooth function of \( z \).

Given the definitions and relationships in (17)–(20), define the implicit profit function in terms of real (deflated) profit, prices, and \( z \) as follows:

(21) \[ H \left( \frac{f(\pi/\alpha(p, z), z) - \beta(p, z)}{\gamma(p, z)} - w(\delta(p, z), z), z \right) = 0, \quad \frac{\partial H(x, z)}{\partial x} > 0. \]

In this representation, netputs are aggregable in deflated heterogeneous profit in the Gorman sense. Appendix A proves the necessity of (21) in a very general context as Proposition 2, our main theoretical result. Sufficiency of (21) is shown by an application of the implicit function theorem,

(22) \[ 0_n = \frac{\partial}{\partial x} H \left( \left( \frac{f - \beta}{\gamma} \right) - w, z \right) \times \frac{1}{\gamma} \left[ f_{x/\alpha} \times \left( \frac{y - \pi \alpha_p}{\alpha^2} \right) - \beta_p \right] - \left( \frac{f - \beta}{\gamma} \right) \frac{\gamma_p}{\gamma} - w, \delta_p \right]. \]

Combining this with (19), (20), and \( \partial H(x, z)/\partial x > 0 \) implies (after a bit of algebra),
\[ y = \left( \frac{\alpha_p}{\alpha} \right) \pi + \left( \frac{\alpha}{f_{\pi/\alpha}} \right) \left\{ \beta_p + \gamma_p \left( \frac{f - \beta}{\gamma} \right) + \gamma \delta_p \left[ \chi + \left( \frac{f - \beta}{\gamma} \right)^2 \right] \right\} \]

(23)

\[ = \left( \frac{\alpha_p}{\alpha} \right) \pi + \alpha \left\{ \beta_p - \left( \frac{\beta}{\gamma} \right) \gamma_p + \gamma \left[ \chi + \left( \frac{\beta}{\gamma} \right)^2 \right] \delta_p \left( \frac{1}{f_{\pi/\alpha}} \right) \right\} \]

\[ + \alpha \left( \frac{\gamma_p}{\gamma} - 2 \beta \delta_p \right) \left( \frac{f}{\pi_p} \right) \left( \frac{f^2}{\pi_p} \right) + \delta_p \left( \frac{f^2}{\pi_p} \right). \]

Thus, netputs are linear in functions of real profit and \( z \), facilitating aggregation.\(^{10}\)

Notice that (23) has up to four independent vector-valued price functions associated with each of up to four functions of profit. In the language of Gorman (1981) and Lewbel (1989, 1990), the rank of any aggregable netput system can be no greater than four. The functions of profit are: \( \{\pi, 1/f_{\pi/\alpha}, f/f_{\pi/\alpha}, f^2/f_{\pi/\alpha}\} \). Note that if \( f \propto \ln \pi, \pi^\kappa, \pi^\tau \), for \( \kappa \) any constant other than zero, \( \tau \) any positive constant, and \( \tau = \sqrt{-1} \), then either \( 1/f_{\pi/\alpha} \) or \( f/f_{\pi/\alpha} \) is proportional to \( \pi \), reducing the maximum number of independent functions of profit to three.

**Empirical Preliminaries**

In our empirical application, we employ a parsimonious representation that nevertheless highlights the issues discussed above and extends current practice. Rank two appears to be sufficient given the aggregate data that we examine. One such form is the PIGL model of Muellbauer (1975) – case \( ii. \) above:

(24)

\[ y = \beta^{-1} \beta_p \tau + \beta^\kappa \alpha_p \tau^{1-\kappa}, \kappa \neq 0. \]

Note that
\[ y_{p'}^i = g_{p'} + g_{p}\pi^\top \]

\[ = \beta^{-1}\beta_{pp}\pi + \left[ \beta^\kappa \alpha_{pp'} + \beta^{\kappa-1}(\alpha_p\beta_{p'} + \beta_{p}\alpha_{p'}) \right] \pi^{1-\kappa} + (1-\kappa)\beta^{2\kappa}\alpha_p\alpha_{p'}\pi^{1-2\kappa}, \]

which is clearly a symmetric matrix. As noted in case ii. above, homogeneity requires that \( \beta \) is \( 1^o \) homogeneous in \( p \) (hence, \( \beta_p, p = \beta \) and \( \beta_{pp}, p = 0 \)), \( \alpha \) is \( 0^o \) homogeneous in \( p \) (hence, \( \alpha_p, p = 0 \) and \( \alpha_{pp}, p = -\alpha_p \)), and therefore, \( (g_{p'} + g_{p}\pi^\top)p = 0 \). With these homogeneity properties, the netputs add up in a complete system, \( p^\top y = \pi \).

Consider including fixed inputs apart from \( \pi \). One simple way to add fixed inputs and preserve adding up and homogeneity is to assume \( \beta(p, z) = h(p)\phi(z) \), so that

\[ y(p, z) = h(p)^{-1}h_p(p)\pi(p, z) + \phi(z)^\kappa h(p)^\kappa\alpha_p(p)\pi(p, z)^{1-\kappa}, \]

where \( h \) is \( 1^o \) homogeneous in \( p \). However, consistent with the common practice of using the translog, one could instead assume that

\[ \beta(p, z)^{-1}\beta_p(p, z) = \Delta\{p_i^{-1}\}\left(\xi + B\ln p + C\ln z\right), \]

where \( \Delta\{p_i^{-1}\} \) is a diagonal matrix with \( p_i^{-1} \) as the \( i^{th} \) main diagonal element. Carefully choosing the function of integration for the partial differential equation system in (27) as

\[ \theta(z) = \tilde{A}\left(\prod_{i=1}^m z_i^{\sigma_i}\right)e^{\frac{1}{2}\ln z^\top D\ln z} \]

then gives

\[ \beta(p, z) = \tilde{A}\left(\prod_{j=1}^n p_j^{\tilde{z}_j}\right)\cdot\left(\prod_{i=1}^m z_i^{\sigma_i}\right)e^{\frac{1}{2}\ln p^\top B\ln p + \ln p^\top C\ln z + \frac{1}{2}\ln z^\top D\ln z}. \]

One way the translog can become a nested model is with \( \alpha_p = 0 \). For example, setting \( \alpha(p) = \sum_{i=1}^n \mu_i p_i \), the null hypothesis for the translog model would be \( H_0 : \mu = 0 \).
A more parsimonious and interesting approach to nest the translog within (24) is to define $\ln \beta = \ln A + \zeta^\top \ln p$, so that $\beta$ is a Cobb-Douglas function of $p$\textsuperscript{11} and to define $\alpha_p = A \left( p_i^{-1} \right) (B \ln p + C \ln z)$. Then, the estimating equations of the PIGL form are:

$$
y = \Delta \left( p_i^{-1} \right) \left[ \zeta \pi + \prod_{i=1}^n p_i^{\kappa_i} (B \ln p + C \ln z) \pi^{1-\kappa} \right].$$

Rank 1, the Cobb-Douglas, is given by $\kappa = 0$ and $B \ln p + C \ln z$ proportional to $\zeta$, which requires $B = [0]_{n \times n}$ and $C = [0]_{n \times m}$. A translog system of netput equations is given by $\kappa = 0$. A parametric alternative hypothesis for rank 2 versus the Cobb-Douglas model is $H_A : \kappa \neq 0, B \neq [0]_{n \times n}, C \neq [0]_{n \times m}$. In (29), homogeneity requires that $\alpha$ is 0\textsuperscript{o} homogeneous in $p$, so that each of the column sums of $B$ and $C$ is zero, and 1\textsuperscript{o} homogeneity of $\beta$ in $p$, so that $\sum_{i=1}^n \zeta_i = 1$. In summary, for one additional parameter, $\kappa$, implicit netputs obtain a substantially greater flexibility than in the translog and allows a nested test for the translog functional form.

**An Application**

One of the most prominent of U.S. agricultural production data sets, summarized in Ball et. al. (2004), is the annual time series for the years 1947-1994 produced and maintained by the Economic Research Service of the U.S. Department of Agriculture. The groupings of netputs are: $y_1 =$ livestock, $y_2 =$ crops, $y_3 =$ chemicals, $y_4 =$ fuels and electricity, $y_5 =$ feed, seed, and livestock purchases, labeled FSL, $y_6 =$ hired labor, and the numeraire and omitted netput, $y_7$, is other purchased inputs. Own price indexes correspond to the netput groupings: for example, $p_1 =$ the price of livestock. Quasi-fixed inputs are: $z_1 =$ operator
labor, $z_2 = \text{capital}$, and $z_3$ is time (exogenous technical change). This is similar in scope to the data analyzed in Antle (1984), but for a different sample period.

With a system of six nonlinear netput equations, even with a parsimonious set of instruments, it is not practical to undertake Newey-West heteroskedasticity and autocorrelation consistent (HAC) estimation procedures. The Newey-West HAC estimator requires the number of orthogonality conditions to exceed the number of structural parameters for the conditional mean (45 or 46 in the current application), but not to exceed the number of time series observations, 47.

Thus, the netput equations are estimated in share form,

$$\pi_t^{-1} \Delta \left\{ p_i \right\} y_t = s_t = \xi + \prod_{i=1}^n p_i^{\kappa} (B \ln p_i + C \ln z_i) \pi_t^{-\kappa} + u_i, \ t = 1, \ldots, 47,$$

or in expenditure form,

$$\Delta \left\{ p_i \right\} y_t = e_t = \omega + \prod_{i=1}^n p_i^{\kappa} (B \ln p_i + C \ln z_i) \pi_t^{1-\kappa} + u_i, \ t = 1, \ldots, 47,$$

with $u_t = Ru_{t-1} + e_t$, $E(e_t e_{t'}) = \Omega$, $E(e_t e_{t'}) = [0]_{n \times n}$, $t \neq t'$, $\forall \ t, t' = 1, \ldots, 47$, and the roots of $R$ imply that the AR(1) process is stable.

Therefore, $E(u_t \mid x_t) = 0$, where $x_t$ represents exogenous variables. This is a first order vector autoregressive scheme studied by Berndt and Savin (1975), Moschini and Moro (1994), and Holt (1998), among others. Adding up over the 7 netputs imposes a structure on $R$. Due to the limited sample size, we report results for a diagonal $R$ with a common autoregressive parameter $\rho$ for each equation. Solving for $e_t$ implies that (30) can be written in quasi-difference form:
\[ e_t = s_t - \rho s_{t-1} = \left[ \zeta + \prod_{i=1}^{n} p_{it}^{\kappa_i} (B \ln p_i + C \ln z_t) \pi_t^{-\kappa} \right] \]
\[ + \rho \left[ \zeta + \prod_{i=1}^{n} p_{it-1}^{\kappa_i} (B \ln p_{i-1} + C \ln z_{t-1}) \pi_{t-1}^{-\kappa} \right], \quad t = 1, \ldots, 47. \]

This kind of economic system is often estimated by (linear or nonlinear) seemingly unrelated regression (SUR) methods (Antle 1984). However, there clearly are potential issues with endogenous regressors, especially with profit on the right hand side of (30) or (31). With aggregate data on an increasingly open economy, it is unclear if prices are endogenous, or how to properly take this into account short of building an international model of demand and supply.

Weather events that affect outputs, for example, surely also affect profit because the same phenomenon causes output and profit. However, profit aggregates the impacts of weather across the different netputs and may tend to cancel these weather effects out. A similar aggregation argument applies to measurement errors in the dependent variables and model specification errors.

We present SUR estimates of (32) and the restricted version with \( \kappa = 0 \) in Table 1 as a contrast to generalized method of moments (GMM) estimates. The results reported in Table 1 are for the parameterization \( \lambda = 1 - \kappa \), instead of \( \kappa \). To facilitate the connection to conventional translog estimates, the model estimates are presented in share form. However, the results are similar in expenditure form.

In Table 1, it is apparent that much is being asked of the data. Slightly less than half of the 45 parameters are statistically different from zero at the .01 significance level, which is typical of large systems using aggregate data. There is clear evidence of autocor-
relation in the error terms (the P-value for $\rho$ is less than .001) with a stable dynamic process ($|\hat{\rho}| < 1$). The Cobb-Douglas, $B = [0]_{n \times n}$, $C = [0]_{n \times m}$, $\lambda = 1$ ($\kappa = 0$), is rejected with a Wald statistic of 237.71, which substantially exceeds the 1% critical value of $\chi^2_{0.01}(40) = 76.15$. A Wald-test of $\zeta = 0$ gives a test static of 109.997 and P-value < .001, implying clear rejection that the intercepts jointly are zero. Further, a Wald-test rejects $B = [0]$ at a .014 level of significance. The SUR estimates imply rejection of the Translog: the point estimate of $\lambda$ is .875 and an asymptotic $t$–test implies that $\lambda$ differs from 1 at the .01 level ($t_{1-\hat{\lambda}} = 2.36$).

The elements of $B$ are not the elasticities in either the translog or the PIGL form (29). The analogue to Marshallian price elasticities in consumer theory can be calculated at the sample means (where inputs are the absolute value of the netputs), for a fixed level of profit, as:

$$
\frac{p_j}{y_i} \frac{\partial y_i}{\partial p_j} = \left\{ \begin{array}{ll}
-1 + \frac{(\bar{s}_i - \bar{c}_i)}{\bar{s}_i} \left[ \kappa \zeta_j + \left( \frac{b_{ij}}{\sum_{k=1}^{n} b_{ik} \ln p_k + \sum_{l=1}^{m} c_{il} \ln z_l} \right) \right], & j = i, \\
(\bar{s}_i - \bar{c}_i) \left[ \kappa \zeta_j + \left( \frac{b_{ij}}{\sum_{k=1}^{n} b_{ik} \ln p_k + \sum_{l=1}^{m} c_{il} \ln z_l} \right) \right], & j \neq i,
\end{array} \right.
$$

where bars denote sample averages. Similarly, the netput elasticities with respect to profit, also calculated at the sample means, are given by:

$$
\frac{\pi}{y_i} \frac{\partial y_i}{\partial \pi} = \frac{\zeta_i}{\bar{s}_i} + (1 - \kappa) \left( \frac{\bar{s}_i - \bar{c}_i}{\bar{s}_i} \right), \quad i = 1, \ldots, n.
$$

The elements of the analogue to Hicksian substitution elasticities are defined by:
Combining (33)–(35) and grouping terms gives the elements of the substitution elasticity matrix at the means of the data as:

\[
\eta_{ij} = \begin{cases} 
-1 + \xi_i + \left( \frac{\bar{s}_i - \bar{s}_j}{\bar{s}_i} \right) \left[ \kappa \xi_j + (1 - \kappa) \bar{s}_i + \left( \frac{b_{ij}}{\sum_{k=1}^{n} b_{jk} \ln p_k + \sum_{l=1}^{m} c_{il} \ln z_l} \right) \right], & j = i, \\
\xi_i \left( \frac{\bar{s}_j - \bar{s}_i}{\bar{s}_i} \right) \left[ \kappa \xi_j + (1 - \kappa) \bar{s}_i + \left( \frac{b_{ij}}{\sum_{k=1}^{n} b_{jk} \ln p_k + \sum_{l=1}^{m} c_{il} \ln z_l} \right) \right], & j \neq i.
\end{cases}
\]

To economize on space for the SUR results, which are not the central focus of the study, only the own-price elasticities are presented in Table 2. Although there is no clear pattern to the bias by imposing the restriction \( \lambda = 1 \), it is clear that the bias is not small, averaging approximately 22.3% and ranging up to 50.5%.

Table 3 and 4 present GMM estimates. The instruments are: the six lagged relative prices, relative prices lagged two periods in the crop and livestock equations, the squares of lagged prices, lagged profit and its square, and the fixed inputs and their square. In addition, a number of macro instruments were investigated: GDP and GDP/capita, t-bill interest rates, population, exports, and exchange rates. Adding these instruments from the macro economy did not significantly change the results reported in Tables 3 and 4 using only lagged instruments from the agricultural sector.

J-tests indicate for both the case of \( \lambda \) unconstrained and \( \lambda = 1 \) (translog), that there is insufficient evidence to reject the orthogonality conditions with P-values of 0.242 and 0.312, respectively. As is often the case for a highly parameterized model, there is evi-
evidence that little more than half of the coefficients are significantly different from zero (26/47 for the unrestricted GMM estimates).

For the translog model, the effects of exogenous technical change (e.g., \( c_{33}, c_{43} \)) often appear to be of the wrong sign. Conventional wisdom suggests that the absolute value of shares for hired labor, chemicals, and energy should increase with time during this time period (Binswanger 1974). In contrast, results for the unrestricted model are consistent with this conventional wisdom.

The \( t \)-test of \( \lambda = 1 \) gives a test statistic of 2.552, which implies a rejection of the null hypothesis at approximately the 0.01 significance level. A Wald test of \( \zeta = 0 \) gives a Chi-square statistic of 12.565 and a P-value of 0.05 (hired labor is likely to be the main reason that the P-value isn’t lower). A test of \( B = [0]_{n \times n} \) gives a statistic of 68.697, indicating a clear rejection of the null hypothesis (P-value < .0001). It seems clear that there is credible evidence for rejecting the rank one case, including the translog model.

Finally, at the sample means, the eigen values of the substitution matrix are 5.52, 0.94, 0.36, 0.07, 0.01, and –0.04, implying a deviation from the hypothesis of convexity of \( \pi \) in \( p \), in so far as one root is negative but very small in absolute value.

Table 4 is analogous to Table 2 but for GMM. There is no clear pattern of differences between the SUR and GMM methods (either more or less elastic). It is apparent that the unconstrained and constrained estimates are again considerably different, although both models have the expected signs for the elasticities. Because livestock in particular should display a dynamic response path, the large difference in the results for livestock using these estimation techniques may not be all that interesting, important, or surprising. Fur-
thermore, the precision of the elasticity estimates is quite low, judging by the estimated asymptotic standard errors. The next largest differences are for feed, seed, and livestock and hired labor. In the first case (FSL), the unrestricted case yields lower elasticities and for hired labor they are larger. Both elasticities for hired labor seem large, with the unconstrained elasticity more so than for the constrained (translog) model. However, the estimated asymptotic standard errors in both cases suggest that the estimation precision is quite high. Finally, for completeness, Table 5 presents the full table of unconstrained GMM elasticities and their estimated standard errors. Diagonal entries already appeared in Table 4. As is apparent from this table, most cross-price elasticities are not significantly different from zero, indicating relatively low estimation precision. Expected signs are generally met with inputs being normal (e.g., positive (negative) entries down the first two columns (rows), omitting the diagonal entries).

Because livestock and crop production do not have the usual substitution relationship based on fixed resources, it seems reasonable that crop production would increase when the price of livestock increases. Further, symmetry predicts that the response of livestock to crop prices to be the same sign as implied in the Table. There is evidence that a rise in energy prices reduces crop output as expected. Regarding inputs, one might expect in aggregate data that inputs will be gross complements (cross elasticities for inputs are negative) due to their being normal. However, there are seemingly some statistically significant anomalies: most having to do with labor. For example, the cross elasticity of labor demand with respect to energy is .435 and is statistically significant at the 5% level. We note that the positive sign also is present in the translog specification. The parameter es-
ticates offer some evidence that energy and hired labor are substitutes, which may be a reasonable substitution response. To summarize, Table 5 seems for the most part economically reasonable, although it highlights the difficulty of obtaining precise estimates of cross-price substitution elasticities in a flexible form using aggregate time-series data.

**Summary and Conclusions**

This paper has advanced the idea of implicit netput functions (which depend on prices, restricted profit, and quasi-fixed inputs) in production economics. Advantages of such functions are that they are parsimonious in parameters and have simple implementations. The primary disadvantage is that netputs are implicit on the right hand side and explicit on the left hand side of a normalized structural system. However, appropriate estimators are readily available to estimate the system (e.g., GMM or nonlinear three stage least squares).

Developments from demand theory are useful for the implicit netput model and give a clear understanding of flexibility and aggregation. Both are related to the rank of a Gorman system of demands. The ubiquitous translog production system is rank one. Opportunities to expand both the rank and flexibility of this model are apparent. An empirical application examines aggregate U.S. agricultural data using a rank two system nesting the translog model. SUR and GMM estimates and tests reject the translog system in favor of a more general rank two PIGL system.
References


Table 1. SUR parameter estimates of U.S. agriculture.†

<table>
<thead>
<tr>
<th>Estimated Parameter</th>
<th>Non-Linear (unconstrained $\lambda$)</th>
<th>Translog ($\lambda = 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.875** (0.038)</td>
<td>0.448** (0.057)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.711** (0.021)</td>
<td>1.986* (0.819)</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>-2.683* (1.102)</td>
<td>0.269 (0.214)</td>
</tr>
<tr>
<td>$b_{11}$</td>
<td>10.620** (2.068)</td>
<td>-0.753** (0.133)</td>
</tr>
<tr>
<td>$b_{12}$</td>
<td>0.358 (1.517)</td>
<td>-0.753** (0.133)</td>
</tr>
<tr>
<td>$b_{13}$</td>
<td>-0.788* (0.374)</td>
<td>0.091* (0.036)</td>
</tr>
<tr>
<td>$b_{14}$</td>
<td>-0.966** (0.268)</td>
<td>0.034 (0.030)</td>
</tr>
<tr>
<td>$b_{15}$</td>
<td>-6.614** (1.444)</td>
<td>0.166 (0.171)</td>
</tr>
<tr>
<td>$b_{16}$</td>
<td>0.1011* (0.395)</td>
<td>0.081* (0.039)</td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>-1.233 (1.334)</td>
<td>0.240 (0.269)</td>
</tr>
<tr>
<td>$c_{12}$</td>
<td>-4.258 (2.729)</td>
<td>0.186 (0.398)</td>
</tr>
<tr>
<td>$c_{13}$</td>
<td>-2.981** (0.745)</td>
<td>-0.040 (0.163)</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>-0.503 (0.589)</td>
<td>3.158** (0.546)</td>
</tr>
<tr>
<td>$b_{22}$</td>
<td>1.272 (0.803)</td>
<td>-0.162 (0.136)</td>
</tr>
<tr>
<td>$b_{23}$</td>
<td>-0.098 (0.269)</td>
<td>0.115** (0.035)</td>
</tr>
<tr>
<td>$b_{24}$</td>
<td>0.184 (0.262)</td>
<td>0.143** (0.026)</td>
</tr>
<tr>
<td>$b_{25}$</td>
<td>-1.228 (1.180)</td>
<td>0.397** (0.131)</td>
</tr>
<tr>
<td>$b_{26}$</td>
<td>-0.479 (0.276)</td>
<td>0.098** (0.033)</td>
</tr>
<tr>
<td>$c_{21}$</td>
<td>0.143 (0.830)</td>
<td>-0.143 (0.171)</td>
</tr>
<tr>
<td>$c_{22}$</td>
<td>1.424 (1.701)</td>
<td>0.790** (0.264)</td>
</tr>
<tr>
<td>$c_{23}$</td>
<td>2.548** (0.754)</td>
<td>-0.168 (0.104)</td>
</tr>
<tr>
<td>$\gamma_3$</td>
<td>0.356* (0.179)</td>
<td>-0.715** (0.142)</td>
</tr>
<tr>
<td>$b_{33}$</td>
<td>-0.131 (0.133)</td>
<td>-0.073** (0.016)</td>
</tr>
<tr>
<td>$b_{34}$</td>
<td>0.116 (0.063)</td>
<td>-0.010 (0.009)</td>
</tr>
<tr>
<td>$b_{35}$</td>
<td>0.751* (0.312)</td>
<td>-0.076* (0.037)</td>
</tr>
<tr>
<td>$b_{36}$</td>
<td>-0.007 (0.106)</td>
<td>-0.040** (0.011)</td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>0.102 (0.246)</td>
<td>0.092* (0.045)</td>
</tr>
<tr>
<td>$c_{12}$</td>
<td>-0.932 (0.542)</td>
<td>-0.226** (0.071)</td>
</tr>
<tr>
<td>$c_{13}$</td>
<td>-0.690** (0.203)</td>
<td>0.059* (0.028)</td>
</tr>
<tr>
<td>$\gamma_4$</td>
<td>0.358* (0.144)</td>
<td>-0.500** (0.120)</td>
</tr>
</tbody>
</table>
### Table 1, Continued.

|   | \( b_{44} \) | \( b_{45} \) | \( b_{46} \) | \( c_{41} \) | \( c_{42} \) | \( c_{43} \) | \( \eta \) | \( b_{55} \) | \( b_{56} \) | \( c_{51} \) | \( c_{52} \) | \( c_{53} \) | \( \xi \) | \( b_{65} \) | \( c_{61} \) | \( c_{62} \) | \( c_{63} \) |
|   | \(-0.291\) (0.149) | \(0.630^{**}\) (0.242) | \(0.256^{**}\) (0.074) | \(-0.203\) (0.176) | \(-0.411\) (0.384) | \(-0.644^{**}\) (0.180) | \(2.556^{**}\) (0.938) | \(3.762^{*}\) (1.602) | \(1.370^{**}\) (0.374) | \(1.599\) (1.177) | \(1.460\) (1.830) | \(-2.572^{**}\) (0.654) | \(0.460^{*}\) (0.190) | \(-0.118\) (0.162) | \(-0.460\) (0.304) | \(0.671\) (0.719) | \(-0.638^{**}\) (0.215) |
|   | \(-0.080^{**}\) (0.001) | \(-0.065^{*}\) (0.028) | \(0.001\) (0.009) | \(-0.014\) (0.039) | \(-0.181^{**}\) (0.060) | \(0.023\) (0.024) | \(-1.422^{*}\) (0.659) | \(-0.405^{*}\) (0.157) | \(-0.022\) (0.036) | \(-0.074\) (0.214) | \(-0.433\) (0.321) | \(0.051\) (0.130) | \(-0.582^{**}\) (0.152) | \(-0.068^{**}\) (0.014) | \(-0.079\) (0.049) | \(-0.077\) (0.075) | \(0.036\) (0.030) |

†Estimates of equation (32), 1947-1994, for aggregate U.S. agriculture. Estimates are obtained using linear (\( \lambda = 1 \)) and nonlinear (\( \lambda \) unconstrained) SUR. Standard errors are given in parentheses beneath estimates. ** and * indicate significance at the 0.01 and 0.05 levels, respectively.
Table 2. SUR estimates of own-price elasticities for aggregate U.S. agriculture.

<table>
<thead>
<tr>
<th></th>
<th>Unconstrained $\lambda$ Elasticity Estimates</th>
<th>Unconstrained $\lambda$ Standard Errors</th>
<th>Translog ($\lambda = 1$) Elasticity Estimates</th>
<th>Translog ($\lambda = 1$) Standard Errors</th>
<th>% Absolute Elasticity Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Livestock</td>
<td>0.138</td>
<td>0.296</td>
<td>0.269</td>
<td>0.214</td>
<td>48.7</td>
</tr>
<tr>
<td>Crops</td>
<td>0.120</td>
<td>0.103</td>
<td>0.123</td>
<td>0.109</td>
<td>2.5</td>
</tr>
<tr>
<td>Chemicals</td>
<td>-0.757**</td>
<td>0.113</td>
<td>-0.611**</td>
<td>0.118</td>
<td>19.3</td>
</tr>
<tr>
<td>Energy</td>
<td>-0.321*</td>
<td>0.129</td>
<td>-0.281**</td>
<td>0.102</td>
<td>12.5</td>
</tr>
<tr>
<td>FSL</td>
<td>-0.592*</td>
<td>0.244</td>
<td>-0.891**</td>
<td>0.269</td>
<td>50.5</td>
</tr>
<tr>
<td>Hired Labor</td>
<td>-0.733**</td>
<td>0.119</td>
<td>-0.736**</td>
<td>0.089</td>
<td>.4</td>
</tr>
</tbody>
</table>

*Statistically significant at the 0.05 level. **Statistically significant at the 0.01 level.
Table 3-GMM parameter estimates of U.S. agriculture.

<table>
<thead>
<tr>
<th>Estimated Parameter</th>
<th>Non-Linear (unconstrained $\lambda$)</th>
<th>Translog ($\lambda = 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.926** (0.029)</td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.814** (0.018)</td>
<td>0.785 ** (0.022)</td>
</tr>
<tr>
<td>$\varphi_1$</td>
<td>-5.828* (2.591)</td>
<td>8.330** (2.176)</td>
</tr>
<tr>
<td>$b_{11}$</td>
<td>16.333** (3.215)</td>
<td>0.155 (0.232)</td>
</tr>
<tr>
<td>$b_{12}$</td>
<td>3.378 (1.814)</td>
<td>-0.820** (0.148)</td>
</tr>
<tr>
<td>$b_{13}$</td>
<td>-3.104 ** (0.847)</td>
<td>0.096* (0.044)</td>
</tr>
<tr>
<td>$b_{14}$</td>
<td>-4.288** (1.104)</td>
<td>0.039 (0.039)</td>
</tr>
<tr>
<td>$b_{15}$</td>
<td>-11.633** (2.977)</td>
<td>0.343 (0.182)</td>
</tr>
<tr>
<td>$b_{16}$</td>
<td>-0.989 (0.699)</td>
<td>0.070 (0.044)</td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>-2.827 (1.481)</td>
<td>-1.701** (0.613)</td>
</tr>
<tr>
<td>$c_{12}$</td>
<td>-4.766* (2.410)</td>
<td>-0.974 (0.724)</td>
</tr>
<tr>
<td>$c_{13}$</td>
<td>2.158** (0.481)</td>
<td>-1.680** (0.539)</td>
</tr>
<tr>
<td>$s_2$</td>
<td>-0.1.808 (1.070)</td>
<td>5.586** (1.400)</td>
</tr>
<tr>
<td>$b_{22}$</td>
<td>2.701** (1.045)</td>
<td>-0.148 (0.164)</td>
</tr>
<tr>
<td>$b_{23}$</td>
<td>-0.935* (0.479)</td>
<td>0.115** (0.043)</td>
</tr>
<tr>
<td>$b_{24}$</td>
<td>-1.161 (0.655)</td>
<td>0.145** (0.038)</td>
</tr>
<tr>
<td>$b_{25}$</td>
<td>-3.985* (1.626)</td>
<td>0.406** (0.154)</td>
</tr>
<tr>
<td>$b_{26}$</td>
<td>-0.106 (0.324)</td>
<td>0.153** (0.039)</td>
</tr>
<tr>
<td>$c_{21}$</td>
<td>-0.754 (0.810)</td>
<td>-0.869* (0.394)</td>
</tr>
<tr>
<td>$c_{22}$</td>
<td>-0.506 (1.646)</td>
<td>0.240** (0.473)</td>
</tr>
<tr>
<td>$c_{23}$</td>
<td>2.024** (0.597)</td>
<td>-0.781* (0.340)</td>
</tr>
<tr>
<td>$s_3$</td>
<td>1.527* (0.643)</td>
<td>-1.371** (0.404)</td>
</tr>
<tr>
<td>$b_{33}$</td>
<td>0.577 (0.304)</td>
<td>-0.086** (0.021)</td>
</tr>
<tr>
<td>$b_{34}$</td>
<td>1.140** (0.308)</td>
<td>-0.024 (0.014)</td>
</tr>
<tr>
<td>$b_{35}$</td>
<td>2.816** (0.695)</td>
<td>-0.060 (0.045)</td>
</tr>
<tr>
<td>$b_{36}$</td>
<td>-0.095 (0.223)</td>
<td>-0.044** (0.015)</td>
</tr>
<tr>
<td>$c_{31}$</td>
<td>0.406 (0.290)</td>
<td>0.275* (0.114)</td>
</tr>
<tr>
<td>$c_{32}$</td>
<td>-1.020 (0.617)</td>
<td>-0.088 (0.141)</td>
</tr>
<tr>
<td>$c_{33}$</td>
<td>-0.853** (0.180)</td>
<td>0.221* (0.099)</td>
</tr>
<tr>
<td>$s_4$</td>
<td>2.089 * (0.866)</td>
<td>-1.129** (0.355)</td>
</tr>
</tbody>
</table>
Table 3, Continued.

|   | $b_{44}$ | 1.346**  
|   |        | (0.511)  
|   | $b_{45}$ | 3.784**  
|   |        | (0.976)  
|   | $b_{46}$ | -0.711**  
|   |        | (0.300)  
|   | $c_{41}$ | 0.010     
|   |        | (0.253)  
|   | $c_{42}$ | -1.199    
|   |        | (0.804)  
|   | $c_{43}$ | -0.998**  
|   |        | (0.190)  
|   | $\delta_3$ | 5.278*  
|   |        | (2.219)  
|   | $b_{55}$ | 8.521**  
|   |        | (3.046)  
|   | $b_{56}$ | 0.705     
|   |        | (0.627)  
|   | $c_{51}$ | 2.689*    
|   |        | (1.317)  
|   | $c_{52}$ | 2.057     
|   |        | (1.664)  
|   | $c_{53}$ | -1.868**  
|   |        | (0.418)  
|   | $\delta_6$ | 0.274    
|   |        | (0.349)  
|   | $b_{66}$ | -0.168    
|   |        | (0.159)  
|   | $c_{61}$ | -1.01     
|   |        | (0.285)  
|   | $c_{62}$ | 1.985*    
|   |        | (0.997)  
|   | $c_{63}$ | -0.314    
|   |        | (0.192)  
|   | $J$-Statistic | 93.761     

Estimates of (32), 1947-1994, U.S. agriculture. Estimates obtained using linear ($\lambda = 1$) and nonlinear ($\lambda$ unconstrained) GMM. Standard errors are given in parentheses beneath estimates. ** and * indicate significance at the 0.01 and 0.05 levels, respectively. Tests of overidentifying restrictions (J-test): P-value = .242 for unrestricted $\lambda$ and .312 for the translog. Hence, the orthogonality conditions are not rejected.
Table 4. GMM Estimates of own-price elasticities for aggregate U.S. agriculture.

<table>
<thead>
<tr>
<th></th>
<th>Unconstrained $\lambda$ Elasticity Estimates</th>
<th>Unconstrained $\lambda$ Standard Errors</th>
<th>Translog ($\lambda = 1$) Elasticity Estimates</th>
<th>Translog ($\lambda = 1$) Standard Errors</th>
<th>% Absolute Elasticity Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Livestock</td>
<td>0.038</td>
<td>0.297</td>
<td>0.155</td>
<td>0.232</td>
<td>307.9</td>
</tr>
<tr>
<td>Crops</td>
<td>0.159</td>
<td>0.151</td>
<td>0.133</td>
<td>0.131</td>
<td>16.4</td>
</tr>
<tr>
<td>Chemicals</td>
<td>-0.550**</td>
<td>0.167</td>
<td>-0.519**</td>
<td>0.155</td>
<td>5.6</td>
</tr>
<tr>
<td>Energy</td>
<td>-0.419</td>
<td>0.472</td>
<td>-0.462*</td>
<td>0.215</td>
<td>10.3</td>
</tr>
<tr>
<td>FSL</td>
<td>-0.349</td>
<td>0.317</td>
<td>-0.598*</td>
<td>0.303</td>
<td>71.3</td>
</tr>
<tr>
<td>Hired Labor</td>
<td>-1.297**</td>
<td>0.202</td>
<td>-0.753**</td>
<td>0.125</td>
<td>41.9</td>
</tr>
</tbody>
</table>

The elasticities are derived from the parameter estimates in Table 3 and equation (36).

*Statistically significant at the .05 level. **Statistically significant at the .01 level.
Table 5. Own and cross price elasticity of inputs and outputs.

<table>
<thead>
<tr>
<th>Price Quantity</th>
<th>Livestock</th>
<th>Crops</th>
<th>Agricultural Chemicals</th>
<th>Energy</th>
<th>FSL</th>
<th>Hired Labor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Livestock</td>
<td>0.038 (0.297)</td>
<td>0.428** (0.118)</td>
<td>0.039 (0.061)</td>
<td>-0.682 (0.391)</td>
<td>-0.107 (0.210)</td>
<td>-0.152* (0.060)</td>
</tr>
<tr>
<td>Crops</td>
<td>0.342** (0.094)</td>
<td>0.159 (0.151)</td>
<td>0.003 (0.037)</td>
<td>-0.100* (0.046)</td>
<td>-0.204 (0.122)</td>
<td>-0.100* (0.050)</td>
</tr>
<tr>
<td>Agricultural Chemicals</td>
<td>-0.284 (0.442)</td>
<td>-0.024 (0.335)</td>
<td>-0.550* (0.167)</td>
<td>0.090 (0.202)</td>
<td>0.285 (0.405)</td>
<td>0.370* (0.166)</td>
</tr>
<tr>
<td>Energy</td>
<td>-0.934 (0.816)</td>
<td>-1.276* (0.590)</td>
<td>0.128 (0.287)</td>
<td>-0.419 (0.472)</td>
<td>0.850 (0.750)</td>
<td>0.712* (0.315)</td>
</tr>
<tr>
<td>FSL</td>
<td>0.184 (0.360)</td>
<td>0.437 (0.261)</td>
<td>0.068 (0.096)</td>
<td>0.142 (0.125)</td>
<td>-0.349 (0.317)</td>
<td>-0.096 (0.093)</td>
</tr>
<tr>
<td>Hired Labor</td>
<td>0.951* (0.377)</td>
<td>0.780* (0.387)</td>
<td>0.322* (0.144)</td>
<td>0.435* (0.193)</td>
<td>-0.351 (0.341)</td>
<td>-1.297** (0.202)</td>
</tr>
</tbody>
</table>

Standard errors are in parenthesis below the parameter estimates.

** and * indicate significance at the .01 and .05 levels, respectively.
Endnotes

Thanks to Daniel Bennett for assistance in this project.

1 A theorist might be concerned about a more precise description of the production set as well. Prices are restricted to the positive orthant to facilitate continuity and smoothness assumptions.

2 An example would be \( \pi(p, z) = \alpha(p) + \beta(p)\pi(p, z) + c(p)f(\pi(p, z)) + \zeta(z) \) with netput equations

\[
y_i(p, z) = \pi_i(p, z) = \frac{\alpha_i(p) + \beta_i(p)\pi_i(p, z) + c_i(p)f(\pi_i(p, z))}{1 - (\beta_i(p) + f'(\pi_i(p, z))c_i(p))}, i = 1, ..., n.
\]

3 We don’t comment here on the curvature of \( \pi \) in \( z \) because it doesn’t relate directly to properties of the netput functions.

4 Technically, \( w^T(\partial y / \partial p^T)w > 0 \) for all \( w \neq 0 \) not in the null-space caused by homogeneity, \( w^T y = 0 \).

5 The vector \( z \) is analogous to demographic variables in consumer theory.

6 Nothing essential is changed by having a transformation of netputs.

7 Ex ante, one wouldn’t expect restricted or variable profit to be negative. However, one can and does observe negative short run variable profit for a variety of dynamic issues. In such a case, some of these forms must be restricted or eliminated. For example, the PIGL forms require integer exponents and the logarithmic forms are problematic.

8 In particular, the full rank 3 representations given here are straightforward extensions of equations (7), (15), and (26), and Proposition 2 in LaFrance and Pope (2009, pp. 89-98) to implicit netput functions and their associated implicit profit functions.

9 Equivalently, define the monotonic transformation of deflated profit, \( f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R} \), by

\[
f(\pi/\alpha(p, z), z) = \beta(p, z) + \gamma(p, z)w(\delta(p, z), z) \forall (p, z, \pi) \in \mathbb{R}^+_n \times \mathbb{R}^m \times \mathbb{R}_+.
\]
For notational convenience, we are suppressing the arguments of the functions $f, \alpha, \beta, \delta, \gamma,$ and $\chi$.

There is no need to include the Cobb-Douglas term involving $z$ because it doesn’t enter into Cobb-Douglas netputs separately from $\pi$.

Although diagonal $R$ is a strong assumption, it appears to parsimoniously eliminate most of the autocorrelation. A rank one asymmetric formulation of autocorrelation would require 5 additional parameters.

A first order Taylor’s series of the share equations around $\kappa = 0$ illustrates that the squares of $z$ and lagged $p$ are directly relevant.
Appendix A

Statement and Proof of the Main Result

**Proposition 2:** We are given a smooth, integrable, multiplicatively separable, and finitely additive system of \( n \) partial differential equations (pdes),

\[
\frac{\partial y(x,s)}{\partial x} = \sum_{k=1}^{K} \alpha_k(x) h_k(y(x,s),s),
\]

\( x \in \mathbb{R}^n, \ s \in \mathbb{R}^r, \ y \in \mathbb{R}, \ \alpha_k : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n. \) and \( h_k : \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}, \ k = 1, \cdots, K. \) Define the matrix \( A(x) = [\alpha_1(x), \cdots, \alpha_K(x)] \). If \( \text{rank}[A(x)] = K \), then \( K \leq 3 \), and a solution to the system of pdes, \( \tilde{y}(x,s) \equiv f(y(x,s),s), \ \frac{\partial f(y,s)}{\partial y} \neq 0, \) exists such that

\[
\frac{\partial \tilde{y}(x,s)}{\partial x} = \begin{cases} 
\tilde{\alpha}_1(x,s) & K = 1, \\
\tilde{\alpha}_1(x,s) + \tilde{\alpha}_2(x,s) \tilde{y}(x,s) & K = 2, \\
\tilde{\alpha}_1(x,s) + \tilde{\alpha}_2(x,s) \tilde{y}(x,s) + \tilde{\alpha}_3(x,s) \tilde{y}(x,s)^2 & K = 3,
\end{cases}
\]

where \( \tilde{\alpha}_k(x,s) = \sum_{\ell=1}^{K} \alpha_\ell(x)d_{\ell k}(s), \ \forall k = 1, \cdots, K, \) for some nonsingular \( K \times K \) matrix of smooth functions, \( D(s) = [d_{k \ell}(s)] \), and the \( n \times K \) matrix,

\[
\tilde{A}(x,s) = [\tilde{\alpha}_1(x,s), \cdots, \tilde{\alpha}_K(x,s)] = A(x)D(s),
\]

has the same rank as \( A(x) \).

We precede the proof of this result with an explanation and interpretation of the main ideas. This result is a consequence of Lie’s theory of transformation groups (Lie, 1880; English translation with commentary in Hermann, 1975). The basic tool of analysis is the theory and structure of Lie algebras on manifolds. In the field of differential topology, the terms \( h_k(y,s) \partial h_l(y,s)/\partial y - h_l(y,s) \partial h_k(y,s)/\partial y, \ k \neq \ell, \) are Jacoby brackets. Appending
the differential operator, $\partial/\partial y$, on the right of a Jacby bracket generates the Lie bracket,
\[ [h_k(y,s) \partial h_k(y,s)/\partial y - h_\ell(y,s) \partial h_\ell(y,s)/\partial y] \partial/\partial y. \]
The $K$ differential operators, $h_k(y,s) \partial/\partial y$, $k = 1, \ldots, K$, form a $K$–dimensional system of vector fields on the real line.

The Lie algebra for these vector fields is the linear vector space spanned by the vector fields. A fundamental result is that the largest Lie algebra on the real line has rank (i.e., dimension) three and the basis \{\partial/\partial y, y \partial/\partial y, y^2 \partial/\partial y\} spans this vector space.\(^1\)

The validity of the proposition hinges on the following three properties: (a) the multiplicatively separable and finitely additive structure; (b) linear independence of the \{\alpha_k\} and the \{h_k\} over the $K$–dimensional constants, i.e., $\forall \mathbf{c} \in \mathbb{R}^K$, $A(x)\mathbf{c} \neq \mathbf{0}$, and $c^T h \neq 0$; and (c) integrability of the system of pdes. Restricting attention to smooth functions simplifies the argument greatly, but this is not an essential property.

The proof proceeds in three stages. First, symmetry is used to show that the maximum rank is three for a full rank system. Second, symmetry is used to derive the polynomial representation for any full rank system. Third, the maximum rank of three is shown for any system of polynomial partial differential equations.

In each full rank case, the implications of symmetry are used to first identify a common structure of and relationships among the \{h_k\} and to obtain the polynomial representation. Given this polynomial structure, the implications of symmetry are then used to

\(^1\) Russell (1983, 1996) and Russell and Farris (1993, 1998) are useful introductions to these concepts and their application to systems of consumer demand equations. Guillemin and Pollack (1974), Hydon (2000), Olver (1993), and Spivak (1999) are helpful references on differential geometry and applications of Lie’s theory to differential equation systems. In the present case, appending the differential operator $\partial/\partial y$ does not add to the intuition, insight, or generality of the results, and is omitted throughout the rest of the paper.
derive the structure of and relationships between the new \( \{ \alpha_k \} \) that result once we have identified the structure of the \( \{ h_k \} \). Third, these results are combined to find the most general complete solution for each full rank case. Classical calculus notation is used in an effort to make the underlying mathematical theory and results more accessible to a wider audience.

**Proof:** The system of pdes in (A.1) is integrable if and only if the 2nd-order cross partial derivatives of \( y \) with respect to \( x \) are symmetric,

\[
\frac{\partial^2 y}{\partial x_i \partial x_j} = \sum_{k=1}^{K} \left( \frac{\partial \alpha_{ik}}{\partial x_j} h_k + \alpha_{ik} \frac{\partial h_k}{\partial y} \sum_{\ell=1}^{K} \alpha_{\ell j} h_\ell \right)
\]

\[
= \sum_{k=1}^{K} \left( \frac{\partial \alpha_{jk}}{\partial x_i} h_k + \alpha_{jk} \frac{\partial h_k}{\partial y} \sum_{\ell=1}^{K} \alpha_{\ell i} h_\ell \right) = \frac{\partial^2 y}{\partial x_i \partial x_j} \quad \forall \ i \neq j.
\]

These can be expressed in terms of \( \frac{1}{2} n(n-1) \) vanishing differences,

\[
0 = \sum_{k=1}^{K} \left( \frac{\partial \alpha_{ik}}{\partial x_j} - \frac{\partial \alpha_{jk}}{\partial x_i} \right) h_k + \sum_{k=1}^{K} \sum_{\ell=1}^{K} \alpha_{ik} \alpha_{\ell j} \left( h_\ell \frac{\partial h_k}{\partial y} - h_k \frac{\partial h_\ell}{\partial y} \right) \quad \forall \ j < i = 2, \ldots, n.
\]

In the double sum on the right, if \( k = \ell \), then the term \( \alpha_{ik} \alpha_{jk} \) is multiplied by the Jacoby bracket, \( h_k \frac{\partial h_k}{\partial y} - h_k \frac{\partial h_k}{\partial y} = 0 \). When \( k \neq \ell \), \( h_\ell \frac{\partial h_k}{\partial y} - h_k \frac{\partial h_\ell}{\partial y} \) appears twice, once multiplied by \( \alpha_{ik} \alpha_{\ell j} \) and once multiplied by \( -\alpha_{ij} \alpha_{jk} \). Therefore, rewrite (A.4) as

\[
0 = \sum_{k=1}^{K} \left( \frac{\partial \alpha_{ik}}{\partial x_j} - \frac{\partial \alpha_{jk}}{\partial x_i} \right) h_k + \sum_{k=2}^{K} \sum_{\ell=1}^{K-1} \left( \alpha_{ik} \alpha_{\ell j} - \alpha_{jk} \alpha_{\ell i} \right) \left( h_\ell \frac{\partial h_k}{\partial y} - h_k \frac{\partial h_\ell}{\partial y} \right) \quad j < i = 2, \ldots, n,
\]

a system of \( \frac{1}{2} n(n-1) \) equations in the \( \frac{1}{2} K(K-1) \) Jacoby brackets \( h_\ell \frac{\partial h_k}{\partial y} - h_k \frac{\partial h_\ell}{\partial y} \).

Now define the matrices
\[ (A.6) \quad \mathbf{B} = \begin{bmatrix} 
\alpha_{22}\alpha_{11} - \alpha_{12}\alpha_{21} & \cdots & \alpha_{2k}\alpha_{1\ell} - \alpha_{1k}\alpha_{2\ell} & \cdots & \alpha_{2K}\alpha_{1K-1} - \alpha_{1K}\alpha_{2K-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{zz}\alpha_{y1} - \alpha_{y2}\alpha_{z1} & \cdots & \alpha_{zz}\alpha_{yj} - \alpha_{yj}\alpha_{zj} & \cdots & \alpha_{zz}\alpha_{yk} - \alpha_{yk}\alpha_{zK-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{n2}\alpha_{n1} - \alpha_{n1}\alpha_{n1} & \cdots & \alpha_{nk}\alpha_{n1,\ell} - \alpha_{n1,\ell}\alpha_{nk} & \cdots & \alpha_{nk}\alpha_{n1,K-1} - \alpha_{n1,K-1}\alpha_{nk} \\
\alpha_{n2}\alpha_{n1} - \alpha_{n1}\alpha_{n1} & \cdots & \alpha_{nk}\alpha_{n1,\ell} - \alpha_{n1,\ell}\alpha_{nk} & \cdots & \alpha_{nk}\alpha_{n1,K-1} - \alpha_{n1,K-1}\alpha_{nk} 
\end{bmatrix} , \]

\[ (A.7) \quad \mathbf{C} = \begin{bmatrix} 
\frac{\partial \alpha_{11}}{\partial \tilde{x}_2} & \frac{\partial \alpha_{12}}{\partial \tilde{x}_1} & \cdots & \frac{\partial \alpha_{1K}}{\partial \tilde{x}_1} \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial \alpha_{\ell1}}{\partial \tilde{x}_j} & \frac{\partial \alpha_{\ell2}}{\partial \tilde{x}_i} & \cdots & \frac{\partial \alpha_{\ellK}}{\partial \tilde{x}_i} \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial \alpha_{n-1,1}}{\partial \tilde{x}_n} & \frac{\partial \alpha_{n-1,1}}{\partial \tilde{x}_{n-1}} & \cdots & \frac{\partial \alpha_{n-1,K}}{\partial \tilde{x}_{n-1}} \\
\end{bmatrix} , \]

and the vectors

\[ (A.8) \quad \mathbf{h} = [h_1 \cdots h_K]^T , \]

\[ (A.9) \quad \tilde{\mathbf{h}} = \begin{bmatrix} 
h_1 \frac{\partial h_2}{\partial \tilde{y}} - h_2 \frac{\partial h_1}{\partial \tilde{y}} & \cdots & h_i \frac{\partial h_{i+1}}{\partial \tilde{y}} - h_{i+1} \frac{\partial h_i}{\partial \tilde{y}} & \cdots & h_{K-1} \frac{\partial h_K}{\partial \tilde{y}} - h_K \frac{\partial h_{K-1}}{\partial \tilde{y}} 
\end{bmatrix}^T . \]

\( \mathbf{B} \) is \( \frac{1}{2}n(n-1) \times \frac{1}{2}K(K-1) \), \( \mathbf{C} \) is \( \frac{1}{2}n(n-1) \times K \), \( \mathbf{h} \) is \( K \times 1 \), and \( \tilde{\mathbf{h}} \) is \( \frac{1}{2}K(K-1) \times 1 \), and the structural implications of symmetry for the \( \{h_k\} \) can be written in matrix notation as

\[ \mathbf{B} \tilde{\mathbf{h}} = \mathbf{C} \mathbf{h} . \]

Premultiply both sides by \( \mathbf{B}^T \) to obtain \( \mathbf{B}^T \mathbf{B} \tilde{\mathbf{h}} = \mathbf{B}^T \mathbf{C} \mathbf{h} \). It can be shown that \( \mathbf{B} \) and \( \mathbf{B}^T \mathbf{B} \) have the same rank as \( \mathbf{A} \) (Hermann 1975: 141). Therefore, because \( \mathbf{B}^T \mathbf{B} \) is \( \frac{1}{2}K(K-1) \times \frac{1}{2}K(K-1) \) and has rank no greater than \( K \), it follows that \( K \leq 3 \). This is the fundamental rank result of Lie when \( \mathbf{A} \) has full column rank (Hermann 1975: 143-146).

When \( \mathbf{A} \) is full rank, \( \mathbf{B}^T \mathbf{B} \) is positive definite, so that \( \tilde{\mathbf{h}} = (\mathbf{B}^T \mathbf{B})^{-1} \mathbf{B}^T \mathbf{C} \mathbf{h} \equiv \mathbf{D} \mathbf{h} \) is the solution for \( \tilde{\mathbf{h}} \) in terms of \( \mathbf{h} \). The differential vector \( \tilde{\mathbf{h}} \) (the set of Jacoby brackets for the
system) and the vector \( h \) depend on \((y, s)\) but not on \(x\), while the matrix \( D \) depends on \(x\) but not on \((y, s)\). It follows that \( D \) is independent of \((x, y, s)\); it is a matrix of absolute constants.

Note that \( B \) is \( \frac{1}{2}n(n-1) \times \frac{1}{2}K(K-1) \), \( C \) is \( \frac{1}{2}n(n-1) \times K \), and \( D \) is \( \frac{1}{2}K(K-1) \times K \).

When \( K = 1 \), \( D \) has zero rows (zero Jacoby brackets), when \( K = 2 \), \( D \) has one row and two columns, and when \( K = 3 \), \( D \) has three rows and three columns. When \( K > 3 \), \( D \) has more rows than columns (more Jacoby brackets than functions \( \{h_k\} \)), and a generalized inverse of \( B^T B \) is required to obtain a subset of the elements of \( \tilde{h} \) in terms of \( h \) because neither \( A \) nor \( B^T B \) has full column rank.

To derive the polynomial representation, each full rank case is addressed in turn. This step is aided by the following intermediate result showing that a change of variables can be applied that maintains the finitely additive and multiplicatively separable structure with one \( \{h_k\} \) the constant function, identically in \((y, s)\).

**Lemma 1:** If the system of pdes in (A.1) is integrable, then a smooth transformation, of the form \( \tilde{y}(x, s) = f(y(x, s), s) \), \( \partial f(y(x, s), s)/\partial y \neq 0 \), exists which satisfies

\[
\frac{\partial \tilde{y}(x, s)}{\partial x} = \alpha_i(x) + \sum_{k=2}^{K} \alpha_k(x) \tilde{h}_k(\tilde{y}(x, s), s).
\]

**Proof:** Linear independence of the \( \{h_k\} \) implies that at least one of them does not vanish over any open neighborhood in \( \mathbb{R} \times \mathbb{R}^s \). Therefore, without loss of generality (WLOG),
let this be the first and define the function \( f : \mathbb{R} \times \mathbb{R}' \rightarrow \mathbb{R} \), by

\[
(A.11) \quad f(y(x,s),s) = \int_0^{y(x,s)} h_t(z,s)^{-1} dz.
\]

Then by Leibnitz’ rule for differentiating integrals, we have

\[
\frac{\partial f(y(x,s),s)}{\partial x} = h_t(y(x,s),s)^{-1} \frac{\partial y(x,s)}{\partial x}
\]

\[
(A.12) \quad = \alpha_1(x) + \sum_{k=2}^{K} \alpha_k(x) h_k(y(x,s),s)/h_t(y(x,s),s)
\]

\[
= \alpha_1(x) + \sum_{k=2}^{K} \alpha_k(x) \tilde{h}_k(f(y(x,s),s),s),
\]

where each \( \tilde{h}_k : \mathbb{R} \times \mathbb{R}' \rightarrow \mathbb{R} \), is defined by

\[
(A.13) \quad \tilde{h}_k(f(y(x,s),s),s) = h_k(Y(f(y(x,s),s),s)/h_t(Y(f(y(x,s),s),s), s), k = 2, \ldots, K,
\]

and \( y \equiv Y(f(y,s),s) \) is the inverse of \( f(y,s) \) with respect to \( y \), which, in turn, exists by the fact that \( \partial f(y,s)/\partial y = h_t(y,s)^{-1} \neq 0 \).

The next part of the proof is the polynomial representation. To reduce the notational clutter, drop the \( \sim \)s and redefine \( y : \mathbb{R}^n \times \mathbb{R}' \rightarrow \mathbb{R} \) so that the system can be written in the simpler form given by,

\[
(1') \quad \frac{\partial y(x,s)}{\partial x} = \alpha_1(x) + \sum_{k=2}^{K} \alpha_k(x) h_k(y(x,s),s).
\]

This new definition for \( y \) is a change of variables that always can be carried out without changing the structure of the system of pdes.

---

\(^2\) Changing signs of any of the \( \{\alpha_k\} \) and the associated \( \{h_k\} \) does not change the sign of the differential equation system. Changing all of the signs of either the \( \{\alpha_k\} \) without changing any of signs of the \( \{h_k\} \), or conversely, only changes the sign of the solution \( y \) up to the arbitrary function of integration, \( c(s) \). Hence, WLOG, assume that \( h_1 > 0 \).
$K=1$:

\[(A.14) \quad \partial y(x, s)/\partial x = \alpha_1(x).\]

This is automatically a zero-order polynomial. It remains to identify the implications of symmetry on the vector-valued function $\alpha_1 : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ and then to find the complete solution to the system of pdes (A.14).

Integrability holds if and only if $\partial^2 y/\partial x \partial x^\top = \partial \alpha_i / \partial x^\top$ is symmetric, which is necessary and sufficient for the existence of $\beta : \mathbb{R}^n \to \mathbb{R}^r$, $\partial \beta(x)/\partial x = \alpha_1(x)$, so that

\[(A.15) \quad \partial y(x, s)/\partial x = \partial \beta(x)/\partial x.\]

Integrating directly gives the solution as

\[(A.16) \quad y(x, s) = \beta(x) + c(s),\]

for some $c : \mathbb{R}^r \to \mathbb{R}$. Hence, in the full rank one case, $D(s) \equiv [1]$, a scalar.

$K=2$:

\[(A.17) \quad \partial y(x, s)/\partial x = \alpha_1(x) + \alpha_2(x)h_2(y(x, s), s).\]

Integrability implies that

\[(A.18) \quad \frac{\partial^2 y}{\partial x \partial x^\top} = \frac{\partial \alpha_1}{\partial x^\top} + \frac{\partial \alpha_2}{\partial x^\top} h_2 + \alpha_2 \left( \alpha_1 + \alpha_2 h_2 \right)^\top \frac{\partial h_2}{\partial y}\]

is symmetric. Rewrite this in the form of a matrix of vanishing differences,

\[(A.19) \quad \left( \frac{\partial \alpha_1}{\partial x^\top} - \frac{\partial \alpha_1}{\partial x} \right) + \left( \frac{\partial \alpha_2}{\partial x^\top} - \frac{\partial \alpha_2}{\partial x} \right) h_2 + \left( \alpha_2 \alpha_1^\top - \alpha_1 \alpha_2^\top \right) \frac{\partial h_2}{\partial y} = 0.\]

\[^3 \text{In general, } \beta \text{ can depend on } s \text{ as well as } x, \text{ although } x \text{ must be additively separable from } s. \text{ WLOG, this can be (and, indeed, is) absorbed into the definition of the arbitrary function of integration, } c(s).\]
Since \( \alpha_1 \) and \( \alpha_2 \) are linearly independent, \( \alpha_2 \neq b(s)\alpha_1 \) for any \( b : \mathbb{R}' \to \mathbb{R} \). Otherwise, the rank of \( A = [\alpha_1, b\alpha_1] \) is one not two. Hence, \( \alpha_1\alpha_2^T \) is not symmetric. Since \( \{1, h_2\} \) are linearly independent, \( \partial h_2 / \partial y \neq 0 \); otherwise, \( h_2(y, s) = c(s) \) for some \( c : \mathbb{R}' \to \mathbb{R} \), which is a contradiction.\(^4\) Premultiply (A.19) by \( \alpha_1^T \), postmultiply by \( \alpha_2 \), and divide the result by \( \alpha_1^T\alpha_1\alpha_2^T\alpha_2 - (\alpha_1^T\alpha_2)^2 > 0 \) (by the Cauchy-Schwarz inequality), so that

\[
\frac{\partial h_2}{\partial y} = \left[ \frac{\alpha_1^T(\partial\alpha_2^T/\partial x - \partial\alpha_1^T/\partial x^T)}{\alpha_1^T\alpha_1\alpha_2^T\alpha_2 - (\alpha_1^T\alpha_2)^2} \right] h_2
\]

\[
= c_1 + c_1 h_2,
\]

where \( c_1 \) and \( c_2 \) are absolute constants and both cannot vanish simultaneously.

If \( c_2 \neq 0 \), the integrating factor \( e^{-c_2y} \) gives the general solution to (A.20) as

\[
h_2(y, s) = -(c_1/c_2) + c_3(s)e^{c_2y},
\]

for some \( c_3 : \mathbb{R}' \to \mathbb{R} \). Plugging this back into (A.20), implies that, identically in \((x, y, s)\),

\[
\left( \frac{\partial \alpha_1}{\partial x^T} - \frac{\partial \alpha_1^T}{\partial x} \right) + \left( \frac{\partial \alpha_2}{\partial x^T} - \frac{\partial \alpha_2^T}{\partial x} \right) [- (c_1/c_2) + c_3 e^{c_2y}] + (\alpha_2\alpha_1^T - \alpha_1\alpha_2^T)c_2c_3 e^{c_2y} = 0.
\]

This requires that \( c_2(s) \equiv 0 \), a contradiction of the linear independence of \( \{1, h_3\} \), because

\[
\partial y(x, s) / \partial x = \alpha_1(x) - \alpha_2(x) c_1 / c_2 \equiv \tilde{\alpha}_1(x)
\]

is a rank one, rather than full rank two, system. Hence, \( c_2 = 0 \), and the complete solution

\[^4\] If \( h_2(y, s) = c(s) \), then define \( \tilde{\alpha}_1(x, s) = \alpha_1(x, s) + c(s)\alpha_2(x, s) \), so that \( \partial y / \partial x = \tilde{\alpha}_1(x, s) \), which is a rank one system, rather than full rank two.
to (A.20) is \( h_2(y,s) = c_1 y + c_4(s) \), for some smooth, but otherwise arbitrary function of integration, \( c_4 : \mathbb{R}^r \to \mathbb{R} \). This implies that

\[
\frac{\partial y(x,s)}{\partial x} = \alpha_1(x) + \alpha_2(x) [c_1 y(x,s) + c_4(s)]
\]

(A.24)

\[
= [\alpha_1(x) + \alpha_2(x)c_4] + \alpha_2(x)c_1 y(x,s).
\]

\[
= \tilde{\alpha}_1(x,s) + \tilde{\alpha}_2(x,s)y(x,s).
\]

Thus, \( \mathbf{D}(s) = \begin{bmatrix} 1 & 0 \\ c_4(s) & c_1 \end{bmatrix} \) with \( c_1 \neq 0 \). Integrability reduces to

\[
\frac{\partial^2 y}{\partial x \partial x'} = \frac{\partial \tilde{\alpha}_1}{\partial x} + \tilde{\alpha}_2 \tilde{\alpha}_1 + \left( \frac{\partial \tilde{\alpha}_2}{\partial x} + \tilde{\alpha}_2 \tilde{\alpha}_2 \right) y = \frac{\partial \tilde{\alpha}_1}{\partial x} + \tilde{\alpha}_2 \frac{\partial \tilde{\alpha}_1}{\partial x'} + \left( \frac{\partial \tilde{\alpha}_2}{\partial x} + \tilde{\alpha}_2 \frac{\partial \tilde{\alpha}_2}{\partial x'} \right) y.
\]

Equating like powers in \( y \), \( \partial \tilde{\alpha}_2/\partial x' \) must be symmetric. This is necessary and sufficient for a function, \( \delta : \mathbb{R}^a \times \mathbb{R}^r \to \mathbb{R} \), to exist such that \( \partial \delta(x,s)/\partial x = \tilde{\alpha}_2(x,s) \).

Substituting \( \partial \delta(x,s)/\partial x \) for \( \tilde{\alpha}_2(x,s) \) in (A.25) and combining terms implies

\[
\frac{\partial \tilde{\alpha}_1}{\partial x'} + \frac{\partial \delta}{\partial x} \tilde{\alpha}_1 = \frac{\partial \tilde{\alpha}_1}{\partial x} + \tilde{\alpha}_1 \frac{\partial \delta}{\partial x'},
\]

(A.26)

so that \( \partial \tilde{\alpha}_1/\partial x' - \tilde{\alpha}_1 \partial \delta/\partial x' \) is symmetric. Applying the integrating factor \( e^{-\delta} \) gives,

\[
\frac{\partial}{\partial x} \left[ y(x,s) e^{-\delta(x,s)} \right] = \left[ \frac{\partial y(x,s)}{\partial x} - y(x,s) \frac{\partial \delta(x,s)}{\partial x} \right] e^{-\delta(x,s)} = \tilde{\alpha}_1(x,s) e^{-\delta(x,s)}.
\]

Partially differentiating a second time with respect to \( x' \) then gives

\[
\frac{\partial^2}{\partial x \partial x'} \left[ y(x,s) e^{-\delta(x,s)} \right] = \left[ \frac{\partial \tilde{\alpha}_1(x,s)}{\partial x'} - \tilde{\alpha}_1(x,s) \frac{\partial \delta(x,s)}{\partial x'} \right] e^{-\delta(x,s)}.
\]

(A.28)

Symmetry of the \( n \times n \) matrix on the right then is necessary and sufficient for the existence of a function, \( \gamma : \mathbb{R}^a \times \mathbb{R}^r \to \mathbb{R} \), which satisfies \( \partial y(x,s)/\partial x = \tilde{\alpha}_1(x,s) e^{-\delta(x,s)} \), so that
\[ y(x,s) = \left[ y(x,s) + c(s) \right] e^{\delta(x,s)}, \]  

for some \( c : \mathbb{R}^r \to \mathbb{R}. \)

\( K = 3: \)

\[ \frac{\partial y(x,s)}{\partial x} = \alpha_1(x) + \alpha_2(x)h_2(y(x,s),s) + \alpha_3(x)h_3(y(x,s),s). \]

Because \( \frac{\partial h_1}{\partial y} = 0, \) \( h_1 \frac{\partial h_2}{\partial y} - h_2 \frac{\partial h_1}{\partial y} = \frac{\partial h_2}{\partial y}, \) and \( h_1 \frac{\partial h_3}{\partial y} - h_3 \frac{\partial h_1}{\partial y} = \frac{\partial h_3}{\partial y}, \) we have

\[ \frac{\partial h_2}{\partial y} = d_{11} + d_{12}h_2 + d_{13}h_3, \]

\[ \frac{\partial h_3}{\partial y} = d_{21} + d_{22}h_2 + d_{23}h_3, \]

\[ h_2 \frac{\partial h_3}{\partial y} - h_3 \frac{\partial h_2}{\partial y} = d_{31} + d_{32}h_2 + d_{33}h_3, \]

where each of the \( d_{ij} \)s are independent of \((x,y,s)\) and cannot vanish simultaneously in any row without contradicting full rank three. The first two equations form a complete system of linear first-order pdes. These would be straightforward to solve if the system were not constrained by the third equation.

Our strategy is first to derive the complete solution to this two-equation system of pdes and then to check for consistency with the third equation. The second step restricts the set of values that the \( d_{ij} \) can assume in an integrable full rank three system. First, differentiate the first equation with respect to \( y \) and substitute out \( \frac{\partial h_3}{\partial y} \) and \( h_3, \)
\[ \frac{\partial^2 h_2}{\partial y^2} = d_{12} \frac{\partial h_2}{\partial y} + d_{13} \frac{\partial h_3}{\partial y} \]
\[ = d_{12} \frac{\partial h_2}{\partial y} + d_{13}(d_{21} + d_{22}h_2 + d_{23}h_3) \]
\[ = d_{13}d_{21} + d_{12} \frac{\partial h_2}{\partial y} + d_{13}d_{22}h_2 + d_{23}(\frac{\partial h_2}{\partial y} - d_{11} - d_{12}h_2) \]
\[ = (d_{12} + d_{23}) \frac{\partial h_2}{\partial y} + (d_{13}d_{22} - d_{12}d_{23})h_2 + (d_{13}d_{21} - d_{11}d_{23}). \]

The homogeneous part is
\[ \frac{\partial^2 h_2}{\partial y^2} - (d_{12} + d_{23}) \frac{\partial h_2}{\partial y} - (d_{13}d_{22} - d_{12}d_{23})h_2 = 0, \]
with characteristic equation
\[ \lambda^2 - (d_{12} + d_{23})\lambda - (d_{13}d_{22} - d_{12}d_{23}) = 0, \]
and characteristic roots
\[ \lambda = \frac{1}{2} \left[ d_{12} + d_{23} \pm \sqrt{(d_{12} + d_{23})^2 + 4(d_{13}d_{22} - d_{12}d_{23})} \right]. \]

If \( \lambda = 0 \) is the only characteristic root, then the general solution to the first and second equations in (A.31), equivalently, to (A.32), is of the form
\[ h_2(y, s) = a_2(s) + b_2(s)y + c_2(s)y^2, \]
\[ h_3(y, s) = a_3(s) + b_3(s)y + c_3(s)y^2, \]
where \( a_k, b_k, c_k : \mathbb{R}^r \to \mathbb{R}, k = 2, 3. \) Define \( D(s) = \begin{bmatrix} 1 & 0 & 0 \\ a_2(s) & b_2(s) & c_2(s) \\ a_3(s) & b_3(s) & c_3(s) \end{bmatrix}, \) so that
\[ \begin{bmatrix} \tilde{a}_1(x, s) \\ \tilde{a}_2(x, s) \\ \tilde{a}_3(x, s) \end{bmatrix} = \begin{bmatrix} a_1(x) & a_2(x) & a_3(x) \\ a_2(x) & a_3(x) & a_1(x) \\ a_3(x) & a_1(x) & a_2(x) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a_2(s) & b_2(s) & c_2(s) \\ a_3(s) & b_3(s) & c_3(s) \end{bmatrix}, \]

hence \( \frac{\partial y}{\partial x} = \tilde{a}_1 + \tilde{a}_2y + \tilde{a}_3y^2, \) and by full rank three, \( |D| = b_2c_3 - b_3c_2 \neq 0. \) The third equation in (A.31) then is, \( h_2 \frac{\partial h_3}{\partial y} - h_3 \frac{\partial h_2}{\partial y} = 2y^2 - y^2 = h_3, \) which is consistent
with the other two equations.

It will now be shown that this is the only possibility. Suppose that the characteristic roots are distinct, so that the general solution to (A.32), and hence to (A.31), is of the form

\begin{align}
A.38
h_2(y, s) &= a_2(s) + b_2(s)e^{\hat{\lambda}_2y} + c_2(s)e^{\hat{\lambda}_2y}, \\
h_3(y, s) &= a_3(s) + b_3(s)e^{\hat{\lambda}_3y} + c_3(s)e^{\hat{\lambda}_3y},
\end{align}

for some \(a_k, b_k, c_k : \mathbb{R}^r \to \mathbb{C}, k = 2, 3\). Define the matrix \(D(s)\) in the same way as before, but with complex conjugates for \(bc\) and for \(bc\) if the roots are complex-valued, i.e., \(\hat{\lambda}_1 = \kappa + \sqrt{-1}\tau\) and \(\hat{\lambda}_2 = \kappa - \sqrt{-1}\tau\). This reduces the differential equation system to

\begin{align}
A.39
\hat{\partial}y/\hat{\partial}x \equiv \hat{\alpha}_1 + \hat{\alpha}_2 e^{\hat{\lambda}_2y} + \hat{\alpha}_3 e^{\hat{\lambda}_3y},
\end{align}

i.e., WLOG, \(h = [1 e^{\hat{\lambda}_1y} e^{\hat{\lambda}_2y}]^\top\). The third equation in (A.31) then is

\begin{align}
A.40
(\hat{\lambda}_2 - \hat{\lambda}_1)e^{(\hat{\lambda}_1 + \hat{\lambda}_2)y} = d_{31} + d_{32}e^{\hat{\lambda}_2y} + d_{33}e^{\hat{\lambda}_3y},
\end{align}

where \(\hat{\lambda}_2 - \hat{\lambda}_1 = \sqrt{(d_{12} + d_{23})^2 + 4(d_{13}d_{22} - d_{12}d_{23})} \neq 0\) and \(\hat{\lambda}_2 + \hat{\lambda}_1 = d_{12} + d_{23} \neq \hat{\lambda}_1 \neq \hat{\lambda}_2\), a contradiction of the linear independence of \(\{1, e^{\hat{\lambda}_1y}, e^{\hat{\lambda}_2y}, e^{(\hat{\lambda}_1 + \hat{\lambda}_2)y}\} \quad \forall \ (\hat{\lambda}_1, \hat{\lambda}_2) \neq (0, 0)\).

Hence, suppose that the roots are equal (and therefore real-valued), but do not vanish, \(\hat{\lambda} = \frac{1}{2}(d_{12} + d_{23}) \neq 0\). Then the general solution to the first two equations in (A.31) is

\begin{align}
A.41
h_2(y, s) &= a_2(s) + b_2(s)e^{\hat{\lambda}y} + c_2(s)ye^{\hat{\lambda}y}, \\
h_3(y, s) &= a_3(s) + b_3(s)e^{\hat{\lambda}y} + c_3(s)ye^{\hat{\lambda}y},
\end{align}

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5 In this (impossible) case, we need to consider \(\mathbb{C} = \{x + \sqrt{-1}y : (x, y) \in \mathbb{R}\}\) because the characteristic roots of the 2nd–order homogeneous differential equation might be complex conjugates.
for some \(a_k, b_k, c_k : \mathbb{R}^r \to \mathbb{R}, k = 2, 3\). Again define \(D(s)\) as above (with real elements), so that
\[(A.42) \quad \partial y/\partial x = \tilde{\alpha}_1 + \tilde{\alpha}_2 e^{\lambda y} + \tilde{\alpha}_3 ye^{\lambda y},\]
and, WLOG, \(h = [1 \ e^{\lambda y} \ ye^{\lambda y}]^\top\). The third equation in (A.31) now is
\[(A.43) \quad e^{2\lambda y} = d_{31} + d_{32} e^{\lambda y} + d_{33} ye^{\lambda y},\]
a contradiction of the linear independence of \(\{1, e^{\lambda y}, ye^{\lambda y}, e^{2\lambda y}\} \quad \forall \lambda \neq 0\). Hence, only a repeated vanishing root is possible, and
\[(A.44) \quad \partial y/\partial x \equiv \alpha_1 + \alpha_2 y + \alpha_3 y^2,\]
where, as before, we drop the \(\sim\)s to reduce the notational clutter. This completes the proof of the polynomial form up to at most a quadratic for any full rank system.

We now turn to the relationships among on the \(\{\alpha_k\}\) and the complete solution to
\[(A.44).\] Symmetry requires
\[(A.45) \quad \begin{aligned} \frac{\partial^2 y}{\partial x \partial x^\top} &= \frac{\partial \alpha_1}{\partial x^\top} + \frac{\partial \alpha_2}{\partial x^\top} y + \frac{\partial \alpha_3}{\partial x^\top} y^2 + (\alpha_2 + 2\alpha_3 y)(\alpha_1 + \alpha_2 y + \alpha_3 y^2)^\top \\ &= \frac{\partial \alpha_1}{\partial x} + \frac{\partial \alpha_2}{\partial x} y + \frac{\partial \alpha_3}{\partial x} y^2 + (\alpha_1 + \alpha_2 y + \alpha_3 y^2)(\alpha_2 + 2\alpha_3 y)^\top. \end{aligned}\]
Write this in terms of vanishing differences and group terms in like powers of \(y\),
\[(A.46) \quad \begin{aligned} \left( \frac{\partial \alpha_1}{\partial x^\top} - \frac{\partial \alpha_1}{\partial x} + \alpha_2 \alpha_1^\top - \alpha_1 \alpha_2^\top \right) + \left( \frac{\partial \alpha_2}{\partial x^\top} - \frac{\partial \alpha_2}{\partial x} + 2(\alpha_3 \alpha_1^\top - \alpha_1 \alpha_3^\top) \right) y \\ + \left( \frac{\partial \alpha_3}{\partial x^\top} - \frac{\partial \alpha_3}{\partial x} + \alpha_3 \alpha_2^\top - \alpha_2 \alpha_3^\top \right) y^2 &= 0. \end{aligned}\]
By continuity, each element of each matrix that premultiplies a power of \(y\) in (A.46) must
vanish. Starting with the terms for $\gamma^2$, this implies

\[(A.47) \quad \frac{\partial \alpha_{3i}}{\partial x_j} - \frac{\partial \alpha_{3j}}{\partial x_i} + \alpha_{3i} \alpha_{2j} - \alpha_{3j} \alpha_{2i} = 0, \quad \forall \ i, j = 1, \ldots, n.\]

Premultiply this by $\alpha_{3k}$ to obtain

\[(A.48) \quad \alpha_{3k} \left( \frac{\partial \alpha_{3i}}{\partial x_j} - \frac{\partial \alpha_{3j}}{\partial x_i} \right) + \alpha_{3k} \alpha_{3i} \alpha_{2j} - \alpha_{3k} \alpha_{3j} \alpha_{2i} = 0, \quad \forall \ i, j, k = 1, \ldots, n.\]

Sequentially interchange the roles of $i, j, k$ and add the three terms together to obtain,

\[(A.49) \quad \alpha_{3k} \left( \frac{\partial \alpha_{3i}}{\partial x_j} - \frac{\partial \alpha_{3j}}{\partial x_i} \right) + \alpha_{3k} \alpha_{3i} \alpha_{2j} - \alpha_{3k} \alpha_{3j} \alpha_{2i} = 0, \quad \forall \ i, j, k = 1, \ldots, n.\]

The last line implies that there exists a function, $\beta : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$, and integrating factor, $\gamma : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}_+$, that satisfy

\[(A.50) \quad \alpha_3(x, s) = \gamma(x, s)^{-1} \partial \beta(x, s) / \partial x\]

(Guillemin and Pollak, 1974: 181). Linear independence of the $\{\alpha_k\}$ implies that $\alpha_3 \alpha_2^T \neq \alpha_2 \alpha_3^T$, in particular, that $\partial \alpha_3 / \partial x^T$ cannot be symmetric; hence $\gamma(x, s)$ cannot be constant with respect to $x$. 
Substituting for $\alpha_3$ in the matrix premultiplying $y^2$ in (A.46) now implies

$$\left(\frac{1}{\gamma} \alpha_2 - \frac{1}{\gamma^2} \frac{\partial \gamma}{\partial \mathbf{x}} \right) \frac{\partial \beta}{\partial \mathbf{x}^T} = \frac{\partial \beta}{\partial \mathbf{x}} \left(\frac{1}{\gamma} \alpha_2 - \frac{1}{\gamma^2} \frac{\partial \gamma}{\partial \mathbf{x}} \right)^T.$$  

(A.51)

Grouping terms with common subscripts, this can be rewritten as

$$\gamma \frac{\partial \alpha_2_1}{\partial \mathbf{x}_i} - \frac{\partial \gamma}{\partial \mathbf{x}_i} \frac{\partial \alpha_2_1}{\partial \mathbf{x}_i} = \frac{\partial \beta}{\partial \mathbf{x}_i} \forall i, j,$$

hence a function $\delta: \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$ exists such that

$$\alpha_2 = \gamma^{-1} \frac{\partial \gamma}{\partial \mathbf{x}} - 2\gamma^{-1} \delta \cdot \frac{\partial \beta}{\partial \mathbf{x}}.$$  

The constant $-2$ is a convenient and innocuous normalization.

Substituting for $\alpha_2$ and $\alpha_3$ in the matrix that premultiplies $y$ in (A.46) and rearranging terms implies

$$\left[ \alpha_1 + \left( \frac{\delta}{\gamma} \frac{\partial \gamma}{\partial \mathbf{x}} - \frac{\partial \delta}{\partial \mathbf{x}} \right) \frac{\partial \beta}{\partial \mathbf{x}} \right] = \frac{\partial \beta}{\partial \mathbf{x}} \left[ \alpha_1 + \left( \frac{\delta}{\gamma} \frac{\partial \gamma}{\partial \mathbf{x}} - \frac{\partial \delta}{\partial \mathbf{x}} \right)^T \right].$$  

(A.54)

Again grouping terms with common subscripts, this can be rewritten as

$$\frac{\alpha_{1i} + (\delta/\gamma) \frac{\partial \gamma}{\partial \mathbf{x}_i} - \frac{\partial \delta}{\partial \mathbf{x}_i}}{\alpha_{1j} + (\delta/\gamma) \frac{\partial \gamma}{\partial \mathbf{x}_j} - \frac{\partial \delta}{\partial \mathbf{x}_j}} = \frac{\partial \beta}{\partial \mathbf{x}_i} \forall i, j,$$

which implies that $\phi: \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$ exists such that

$$\alpha_1 = -\left(\delta/\gamma\right) \frac{\partial \gamma}{\partial \mathbf{x}} + \frac{\partial \delta}{\partial \mathbf{x}} + \phi \frac{\partial \beta}{\partial \mathbf{x}}.$$  

Substituting for $\alpha_1$, $\alpha_2$, and $\alpha_3$ in the first matrix in parentheses in (A.46) then implies
\[
\frac{1}{\gamma} \frac{\partial \phi}{\partial x} - \left( \frac{\phi}{\gamma^2} \right) \frac{\partial \gamma}{\partial x} + 2 \left( \frac{\delta^2}{\gamma^3} \right) \frac{\partial \gamma}{\partial x} - 2 \left( \frac{\delta}{\gamma} \right) \frac{\partial \delta}{\partial x} \frac{\partial \beta}{\partial x} = \]

(A.57)

\[
\frac{\partial \beta}{\partial x} \left[ \frac{1}{\gamma} \frac{\partial \phi}{\partial x} - \left( \frac{\phi}{\gamma^2} \right) \frac{\partial \gamma}{\partial x} + 2 \left( \frac{\delta^2}{\gamma^3} \right) \frac{\partial \gamma}{\partial x} - 2 \left( \frac{\delta}{\gamma} \right) \frac{\partial \delta}{\partial x} \right] = \]

Once again grouping terms with common subscripts, this can be rewritten as

\[
\frac{\partial}{\partial x_i} \left[ \left( \frac{\phi}{\gamma} \right) - \left( \delta/\gamma \right)^2 \right] = \frac{1}{\gamma} \frac{\partial \phi}{\partial x_i} - \left( \frac{\phi}{\gamma^2} \right) \frac{\partial \gamma}{\partial x_i} + 2 \left( \frac{\delta^2}{\gamma^3} \right) \frac{\partial \gamma}{\partial x_i} - 2 \left( \frac{\delta}{\gamma^2} \right) \frac{\partial \delta}{\partial x_i} \]

(A.58)

\[
\frac{\partial}{\partial x_j} \left[ \left( \frac{\phi}{\gamma} \right) - \left( \delta/\gamma \right)^2 \right] = \frac{1}{\gamma} \frac{\partial \phi}{\partial x_j} - \left( \frac{\phi}{\gamma^2} \right) \frac{\partial \gamma}{\partial x_j} + 2 \left( \frac{\delta^2}{\gamma^3} \right) \frac{\partial \gamma}{\partial x_j} - 2 \left( \frac{\delta}{\gamma^2} \right) \frac{\partial \delta}{\partial x_j} \]

This implies that a function, \( \theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), exists such that

(A.59)

\[
\phi(x,s) = \delta(x,s)^2/\gamma(x,s) + \gamma(x,s)\theta(\beta(x,s),s),
\]

(Burkil and Burkil, 1970: 230; Goldman and Uzawa, 1964: Lemma 1). Substituting this for \( \phi \) on the right-hand side of \( \alpha_i \) above then gives

(A.60)

\[
\alpha_i = -\left( \delta/\gamma \right) \frac{\partial \gamma}{\partial x} + \frac{\partial \delta}{\partial x} + \left[ \frac{\delta^2}{\gamma} + \gamma \theta \right] \frac{\partial \beta}{\partial x}.
\]

Substituting for \( \{\alpha_1, \alpha_2, \alpha_3\} \) in (A.44) and rearranging terms gives,

\[
\frac{\partial y}{\partial x} = -\frac{\delta}{\gamma} \frac{\partial \gamma}{\partial x} + \frac{\partial \delta}{\partial x} + \left( \frac{\delta^2}{\gamma} + \gamma \theta \right) \frac{\partial \beta}{\partial x} + \left( \frac{1}{\gamma^2} - \frac{2 \delta}{\gamma} \frac{\partial \beta}{\partial x} \right) y + \frac{1}{\gamma} \frac{\partial \beta}{\partial x} y^2
\]

(A.61)

\[
= \frac{\partial \delta}{\partial x} + \left( \frac{y - \delta}{\gamma} \right) \frac{\partial \gamma}{\partial x} + \gamma \left[ \theta + \left( \frac{y^2 - 2\delta^2}{\gamma^2} \right) \right] \frac{\partial \beta}{\partial x},
\]

which reduces further to
This system of Ricatti pdes identifies the structure of all full rank three systems with the multiplicatively separable and finitely additive structure in equation (A.1) above. The complete solution to this pde system is facilitated by a change of variables to

\[(A.63) \quad (,) (,),(,) \text{ so that (A.64)} \quad 2(,) (,) ( (,) ,) (,) . \]

Partially differentiating with respect to \( x^T \) implies

\[(A.65) \quad \frac{\partial^2 z}{\partial x \partial x^T} = (\theta + z^2) \frac{\partial^2 \beta}{\partial x \partial x^T} + \frac{\partial \theta}{\partial \beta} \frac{\partial \beta}{\partial x} \frac{\partial \beta}{\partial x^T} + 2z \frac{\partial \beta}{\partial x} \frac{\partial z}{\partial x^T}. \]

The first two matrices on the right of the equals sign are symmetric, which implies that \((\partial \beta / \partial x)(\partial z / \partial x)^T\) must be symmetric.\(^6\) This implies that a function \( w: \mathbb{R} \times \mathbb{R}^r \to \mathbb{R} \) exists such that \( z(x,s) = w(\beta(x,s),s) \) and

\[(A.66) \quad \partial w(t,s)/\partial t = \theta(t,s) + w(t,s)^2 \forall (t,s) \in \mathbb{R} \times \mathbb{R}^r. \]

Define \( w \) implicitly by the integral,\(^7\)

\[\text{\footnotesize (A.62)} \quad \frac{\partial}{\partial x} \left( \frac{y(x,s) - \delta(x,s)}{\gamma(x,s)} \right) = \left[ \theta(\beta(x,s),s) + \left( \frac{y(x,s) - \delta(x,s)}{\gamma(x,s)} \right)^2 \right] \frac{\partial \beta(x,s)}{\partial x}. \]
(A.67) \[ w(\beta(x), s) = c(s) + \int_0^{\beta(x,s)} \left[ \theta(t, s) + w(t, s)^2 \right] dt, \]
with initial condition \( w(0, s) = c(s) \) for some \( c : \mathbb{R}^r \rightarrow \mathbb{R} \), and terminal condition,

(A.68) \[ w(\beta(x, s), s) = [\gamma(x, s) - \delta(x, s)]/\gamma(x, s). \]

Then the complete (implicit) solution in the full rank three case is given by

(A.69) \[ \gamma(x, s) = \gamma(x, s)w(\beta(x, s), s) + \delta(x, s). \]

References


Olver, P.J. 1993. Applications of Lie Groups to Differential Equations, Second edition,

that \( w \) is smooth \( \forall (t, s) \in \mathbb{R} \times \mathbb{R}^r \). A linear, second-order differential equation with non-constant coefficients generally does not have a simple solution. However, convergent infinite series of simple functions can be found in many cases (Boyce and DiPrima, 1977, chapter 4).
New York: Springer-Verlag.


Appendix B

Algebra for Fixed and Variable Profit Price Elasticities

Equation (29) for the $i^{th}$ netput (note that $\kappa = 1 - \lambda$):

\[(B.1) \quad y_i = p_i^{-1} \left[ \xi_i \pi + \left( \prod_{j=1}^{n} p_j^{\kappa_j} \right) \left( \sum_{j=1}^{n} b_{ij} \ln p_j + \sum_{l=1}^{m} c_{il} \ln z_l \right) \pi^{1-\kappa} \right].\]

Fixed profit own-price partial derivative:

\[
\left. \frac{\partial y_i}{\partial p_i} \right|_{\pi} = -p_i^{-2} \left[ \xi_i \pi + \left( \prod_{j=1}^{n} p_j^{\kappa_j} \right) \left( \sum_{j=1}^{n} b_{ij} \ln p_j + \sum_{l=1}^{m} c_{il} \ln z_l \right) \pi^{1-\kappa} \right]
\]

\[\quad + p_i^{-2} \kappa_i \left( \prod_{j=1}^{n} p_j^{\kappa_j} \right) \left( \sum_{j=1}^{n} b_{ij} \ln p_j + \sum_{l=1}^{m} c_{il} \ln z_l \right) \pi^{1-\kappa}
\]

\[\quad = -p_i^{-1} y_i + p_i^{-1} (y_i - p_i^{-1} \xi_i) \frac{b_{ii}}{\left( \Sigma_k b_{ik} \ln p_k + \Sigma_l c_{il} \ln z_l \right)} \left( \sum_{k} b_{ik} \ln p_k + \sum_{l} c_{il} \ln z_l \right).
\]

Fixed profit own-price elasticity (the analogue to Marshallian own-price elasticities in the theory of consumer demand):

\[
\left. \frac{p_i \frac{\partial y_i}{\partial p_i}}{y_i \frac{\partial p_i}{\partial p_i}} \right|_{\pi} = -1 + \left( \frac{p_i y_i / \pi - \xi_i}{p_i y_i / \pi} \right) \left[ \kappa_i \pi + \frac{b_{ii}}{\left( \Sigma_k b_{ik} \ln p_k + \Sigma_l c_{il} \ln z_l \right)} \right]
\]

\[\quad = -1 + \left( \frac{s_{i} - \xi_i}{s_{i}} \right) \left[ \kappa_i \pi + \frac{b_{ii}}{\left( \Sigma_k b_{ik} \ln p_k + \Sigma_l c_{il} \ln z_l \right)} \right].
\]
Fixed profit cross-price partial derivative:

\[
\left. \frac{\partial y_i}{\partial p_j} \right|_\pi = \kappa \xi_j \left( \frac{p_j^{-1}(y_i - p_i^{-1} \xi_i\pi)}{p_j^{-1}(\prod_{j=1}^n p_j^{k_j} \left( \sum_{j=1}^n b_{ij} \ln p_j + \sum_{l=1}^m c_{il} \ln z_l \right) \pi^{1-\kappa}} \right)
\]

\[
\text{Fixed profit cross-price elasticity (the analogue to Marshallian cross-price elasticities in consumer demand theory):}
\]

\[
\left. \frac{p_j^i \frac{\partial y_i}{y_i}}{\partial p_j} \right|_\pi = \left( \frac{p_j y_i / \pi - \xi_i}{p_j y_i / \pi} \right) \left[ \kappa \xi_j + \left( \frac{b_{ij}}{\sum_k b_{ik} \ln p_k + \sum c_{il} \ln z_l} \right) \right]
\]

\[
\text{Partial derivative with respect to profit:}
\]

\[
\frac{\partial y_i}{\partial \pi} = p_i^{-1} \xi_i + (1 - \kappa) p_i^{-1} \left( \prod_{j=1}^n p_j^{k_j} \left( \sum_{j=1}^n b_{ij} \ln p_j + \sum_{l=1}^m c_{il} \ln z_l \right) \pi^{-\kappa} \right)
\]

\[
= p_i^{-1} \xi_i + (1 - \kappa) \left( y_i - p_i^{-1} \xi_i \pi / \pi \right)
\]

\[
\text{Profit elasticity (the analogue to the income elasticity in consumer demand theory):}
\]

\[
\frac{\pi \frac{\partial y_i}{y_i}}{\partial \pi} = \frac{\xi_i}{s_i} + (1 - \kappa) \left( \frac{s_i - \xi_i}{s_i} \right).
\]
Variable profit price elasticities (the analogue to Hicksian substitution elasticities in consumer demand theory):

\[
\frac{p_j}{y_i} \left( \frac{\partial y_i}{\partial p_j} \right) + \frac{\partial y_i}{\partial \pi} \frac{\partial \pi}{\partial p_j} = \frac{p_j}{y_i} \left( \frac{\partial y_i}{\partial p_j} \right) + \frac{p_j y_j}{y_i} \pi \frac{\partial y_i}{\partial \pi} = \frac{p_j}{y_i} \left( \frac{\partial y_i}{\partial p_j} \right) + s_j \frac{\pi}{y_i} \frac{\partial y_i}{\partial \pi}.
\]

Variable profit own-price elasticity:

\[
\eta_i = -1 + \left( \frac{s_i - \xi_i}{s_i} \right) \left[ \kappa \xi_i + \left( \frac{b_i}{\sum_k b_{ik} \ln p_k + \sum_l c_{il} \ln z_l} \right) \right] + s_i \left[ \frac{\xi_i}{s_i} + (1 - \kappa) \left( \frac{s_i - \xi_i}{s_i} \right) \right],
\]

(B.9)

Variable profit cross-price elasticity:

\[
\eta_{ij} = \left( \frac{s_i - \xi_i}{s_i} \right) \left[ \kappa \xi_j + \left( \frac{b_{ij}}{\sum_k b_{ik} \ln p_k + \sum_l c_{il} \ln z_l} \right) \right] + s_j \left[ \frac{\xi_i}{s_i} + (1 - \kappa) \left( \frac{s_i - \xi_i}{s_i} \right) \right],
\]

\[
= \frac{s_j}{s_i} \left( \frac{s_i - \xi_i}{s_i} \right) \left[ \kappa \xi_j + \left( \frac{b_{ij}}{\sum_k b_{ik} \ln p_k + \sum_l c_{il} \ln z_l} \right) \right] + s_j \left[ \frac{\xi_i}{s_i} + (1 - \kappa) \left( \frac{s_i - \xi_i}{s_i} \right) \right], \quad i \neq j.
\]

(B.10)